Remain:
Let's go back and think about a two level system.
In class before we exactly solve the dynamics when
the system was subjected to a potential $h \alpha \hat{E} + \exp(2i \omega t)$
We solved it going to a rotating frame.
The solution we found for the case when at $t=0$ the atom was in the ground state was

$$P_\phi(t) = |C_\phi(t)|^2 = \frac{R^2}{s^2 + \delta^2} \sin^2 \left( \frac{\delta t}{2} \right) \left( \frac{1}{\omega} \right)$$

with $\delta = \omega - \omega_0$
At very long times compared to $\Delta t_{\text{eff}} = \sqrt{s^2 + \delta^2}$,
we can take the time average of $\sin^2(\frac{\delta t}{2})$
The time average gives: \frac{1}{2} \left( \text{Note } \sin^2(x) = 1 - \cos^2(2x) \right)$
and the average over $\cos$ vanishes

$$P_\phi \sim \frac{1}{2} \frac{R^2}{s^2 + \delta^2} \text{ Lorentzian, FWHM } \sim \delta$$

Similar will happen in a continuum system.

The other point is that at short times
$P_\phi \propto t^2$, not $t$.

What is the issue? Contradicts the Fermi-Golden...
There is no cause for alarm. In realistic situations when the Fermi golden rule is applicable, there is a group of final states, all with nearly the same energy. In other words, the final state forms a continuous set of accessible levels. In these cases, the transition probability needs to be summed over all final states. The linear in \( t \) is a consequence that the transition probability is proportional to the area under the peak. The height scales as \( t^2 \) and width as \( 1/t \). The area under it is \( t \).

Application of the Fermi Golden Rule to problems with final state continua:

Suppose the final states \( |m\rangle \) of the form a continuum as they do in the case of the photoionization process:

\[
\hbar w + H(1s) \rightarrow p + e^{-}
\]

At extremely high proton energies (\( \hbar w \)) one might think of the final state \( e^{-} \) as a free particle and ignore the \( p-e \) interaction.

Two possible ways to treat this problem using Fermi's golden rule:

1. **Box normalization followed by density of states \( g(E) \) analysis**
2. **Energy Normalization**

   1. **Box normalization:** Model the free particle as a particle in a huge
cubic box of size $L$. Let's use periodic boundary conditions for simplicity.

The discrete box eigenstates, normalized to unity, are:

$$\Phi \left( \vec{r} \right) = \frac{1}{L^{3/2}} e^{i \vec{p} \cdot \vec{r}}$$

where the allowed wave-vectors are

$$\vec{p} = \left( n_x \hat{x} + n_y \hat{y} + n_z \hat{z} \right) \frac{2\pi}{L}$$

These states are orthonormal and complete, i.e.,

$$\int \Phi^{*} \left( \vec{r} \right) \Phi \left( \vec{r} \right) d^3r = \delta_{n_x n'_x} \delta_{n_y n'_y} \delta_{n_z n'_z}$$

and

$$\sum_{\vec{p}} \Phi^{*} \left( \vec{r} \right) \Phi \left( \vec{r} \right) = \delta \left( \vec{r} - \vec{r}' \right)$$

To apply the Golden Rule, we normally want to sum over all final states near the energy of interest where $\delta \left( E_f - E_0 - \hbar \omega \right)$ peaks.

We need to calculate expressions like this one:

$$\sum_{n_x,n_y,n_z} F(n_x,n_y,n_z) \frac{dn_x dn_y dn_z}{L^3} \rightarrow \int d^3n \ F(\vec{n})$$

$$= \int d^3n \ F(\vec{p}) \frac{dn_x dn_y dn_z}{dn_x dn_y dn_z}$$

$$= \left( \frac{L}{2\pi} \right)^3 \int d^3n \ F(\vec{p}) \quad \text{or regarding } F \text{ as a function of momentum}$$

$$\vec{p} = \hbar \vec{n} \quad \text{just}$$

$$= \left( \frac{L}{2\pi} \right)^3 \int d^3p \ F(p)$$
We now want to separate the energy integral (in all Golden Rule problems) so we can integrate over \( \int dE_f \delta (E_f - E_0 - \hbar \omega) \rightarrow 1 \)

So consider the sum of rates to all final states near \( E_f = E_0 + \hbar \omega \)

\[
R_{E_0 \rightarrow E_f} = \int d^3n (\frac{2\pi}{\hbar})^3 |\langle \tilde{p}_n | \tilde{V} | 10 \rangle|^2 \delta (E_f - E_0 - \hbar \omega)
\]

\[= \left( \frac{L}{2\pi\hbar} \right)^3 \int d^3p_f (\frac{2\pi}{\hbar}) \left| \langle \tilde{p}_f | \tilde{V} | 10 \rangle \right|^2 \delta (E_f - E_0 - \hbar \omega) \]

Let's use spherical coordinates in momentum space

\[
R_{E_0 \rightarrow E_f} = \left( \frac{L}{2\pi\hbar} \right)^3 \int d\Omega_f \int p_f^2 dp_f \frac{dp_f}{2\pi} \left| \langle \tilde{p}_f | \tilde{V} | 10 \rangle \right|^2 \delta (E_f - E_0 - \hbar \omega)
\]

The quantity \( \left[ \frac{d\Omega_f}{dE_f} p_f^2 \frac{dp_f}{dE_f} \left( \frac{L^3}{2\pi\hbar^3} \right) \right] \equiv \mathcal{G}_f \)

is called the density of final states

The number of states per unit range of \( E_f \)

The Fermi Golden's rule is often written as

\[
dR_{E_0 \rightarrow E_f} = \frac{2\pi}{\hbar} \left| \langle \tilde{p}_f | \tilde{V} | 10 \rangle \right|^2 \mathcal{G}_f \quad \text{To be evaluated at } E_f = E_0 + \hbar \omega
\]

Method 2: Energy Normalization (an alternative)
- Allows us to bypass the calculation of density states factors. For that consider a differently normalized free-particle states:

\( \psi(\mathbf{r}) = \frac{1}{L^{3/2}} e^{-\frac{\mathbf{r} \cdot \mathbf{p}}{\hbar}} \), unity-normalized in a box

\( \psi(\mathbf{r}) = \frac{1}{(2\pi \hbar)^{3/2}} e^{-\frac{\mathbf{r} \cdot \mathbf{p}}{\hbar}} \), Dirac-delta normalized in momentum vector

Finally Dirac-delta normalized in energy

Energy-normalized

\[ \delta\left(\mathbf{p} - \mathbf{p}'\right) = \frac{\delta(E - E')}{p^2} \frac{dE}{dp} \]

\[ U_{E',\mathbf{p}}(\mathbf{r}) = \left( \frac{p^2}{dE} \frac{dE}{dp} \right)^{1/2} U_{\mathbf{p}}(\mathbf{r}) = \left( \frac{1}{(2\pi \hbar)^3} \frac{p^2}{dE} \frac{dE}{dp} \right)^{1/2} \psi(p) \]

Conclusion: The density of states factor (per unit solid angle) \( \frac{d\Omega}{d\Omega} \) if one uses energy normalized final state wave factors. The form of the Fermi-Golden rule becomes

\[ dR_{\mathbf{e}_o - \mathbf{e}_f} = d\Omega_f \frac{2\pi}{\hbar} |\langle \mathbf{u}_{\mathbf{e}_o} | \mathbf{v}_{\mathbf{e}_f} \rangle|^2 \]