Method of Steepest Descent (saddle point method) or Stationary Phase

Consider a general function of \( z \), \( \mathcal{I}(z) \), with a contour integral representation of the form:

\[
\mathcal{I}(z) = \int \underbrace{g(t) \ e^{\frac{z^+ t}{c}}}_{\text{complex}} \ dt \quad \text{with } z, t \text{ complex}
\]

(e.g., this form arises for Laplace, Fourier transforms) "smooth function"

We will assume that the integrand goes to zero at the ends of the contour.

Goal: Find an approximate expression for this integral, valid for \( \mathcal{I}(z) \) at \( |z| \to \infty \)

Nature of \( e^z \) : \( \text{Im}[e^z] \)

\( z^+ t \) oscillates along the imaginary axis, but blows up along the real axis, and decays exponentially along the neg. real axis.

\( e^{z^+ t} = e^{\text{Re}[z^+ t] + i \text{Im}[z^+ t]} \)
For a general complex value of $z$, write

$$z = R(\cos \theta + i \sin \theta) = R e^{i \theta}$$

$$\Rightarrow \text{Im}[z + f(t)] = R \sin \theta f(t), \quad f(t) \text{ is real}$$

So, qualitatively, in the function

$$\exp \left( i \frac{\text{Im}[z + f(t)]}{\|z\|} \right)$$

this can be seen as being the "frequency" of oscillation, and as $|z|$ increases, this function oscillates faster and faster.

\[\text{plot}\]

$$\text{i.e., } \text{Re}\left[\exp \left( i \frac{\text{Im}(z + f(t))}{|z|} \right) \right]$$

\[\text{larger } |z|\]

Such fast oscillations are bad because they imply that there will be challenging cancellation effects when you perform a numerical integration.
Qualitative issues

We expect that the integral will be dominated by regions along \( C \) where \( \Re[z + u(t)] = \text{maximum} \).

**THEOREM**

However, a complex, analytic function can have no maxima or minima inside a finite region of the complex plane.

Call \( t = x + iy \).

Why? Because \( \nabla^2 u(t) = 0 = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \) (where here \( u(t) = z + f(t) \)).

If we suppose that \( f'(t_0) = 0 \), then if we have a maximum at \( \frac{\partial}{\partial x} z + f(t) \) at \( t_0 \) when we approach it along \( x \), \( u \) i.e. \( \frac{\partial^2 u}{\partial x^2} < 0 \).

Then in order for \( \nabla^2 u \) to be zero, \( u \)

\[ \frac{\partial^2 u}{\partial y^2} > 0 \]

\( \rightarrow \) minimum when \( t_0 \) is approached along the \( y \)-axis!
And we have 2 basic needs for this contour:

1. \( \text{Re} \left[ z + f(t) \right] \) is a local maximum along the contour path, i.e., \( z + f'(t_0) = 0 \) and we must find the critical point(s) where this is satisfied.

2. Want to keep \( \text{Im} \left[ z + f(t) \right] \) constant versus \( t \) along the contour, near the point(s) to identify from \( f'(t_0) = 0 \).

\[ \text{Im}(z + f(t)) + \text{Re}(z + f(t)) \]

\[ \Rightarrow \int(z) = \int_C g(t) e^{i \text{Re}[z + f(t)]} dt \]

will be dominated by such regions where the oscillations get "slow" i.e., where the phase is approximately stationary.

Then, if \( C \) passes through such a region near \( t = t_0 \), we can approximately take \( i \text{Im}[z + f(t_0)] \) outside the integral since it is roughly constant and also \( g(t_0) \) since it is "smooth".

\[ \Rightarrow \int(z) = g(t_0) \exp(i \text{Im}[z + f(t_0)]) \int_C e^{i \text{Re}[z + f(t)]} dt \]

near to
So if \( \text{Re}(z + t) \neq \text{maximum} \) at the ends of the contour, we want to search for obeying:

\[
z + f(z) = 0 \quad \text{or} \quad f'(z) = 0 \quad \Rightarrow \quad \frac{df}{dt} \bigg|_{z_0}
\]

Once we have found such a \( z_0 \), we can expand:

\[
f(t) = f(z_0) + f'(z_0)(t - z_0) + \frac{f''(z_0)}{2}(t - z_0)^2 + O((t - z_0)^3)
\]

\[
f(z) = f(z_0) + \frac{1}{2} f''(z_0)(t - z_0)^2 + O((t - z_0)^3)
\]

Then \( e^{i \text{Im}[z + f(t)]} = e^{i \text{Im}(z + f(z_0))} \) and

\[
\text{Re}(z + f(t)) = \text{Re}(z + f(z_0)) + \frac{1}{2} \text{Re}(z + f(z_0)) (t - z_0)^2
\]

\[
\text{e} \quad \frac{\text{Re}(z + f(z_0))}{e} \quad \frac{1}{2} \text{Re}(z + f(z_0)) (t - z_0)^2
\]
To calculate \( \mathcal{I}(z) = \int_C g(t) e^{zt} \, dt \)

at \( |z| \to \infty \),

expand \( f(t) \) about a critical point \( t_0 \),

found by solving

\[ f'(t_0) = 0 \]

\[ \implies f(t) \approx f(t_0) + \frac{1}{2} f''(t_0) (t - t_0)^2, \text{ near } t_0 \]

Next, deform contour \( C \) to pass through \( t_0 \),

at an angle \( \alpha \) to be determined by requiring the integrand to decrease as steeply as possible (steepest descent). That is, set

\[ t - t_0 = \zeta e^{i\phi} \]

with \( \{ \phi = \text{constant} \} \quad \zeta = \text{real} \)

To find \( \alpha \), set \( z = |z| e^{i\phi} \), \( \phi_e = \text{arg}(z) \)

and set \( f''(t_0) = |f''(t_0)| \exp(i\phi_{f''}) \)

\[ \phi_{f''} = \text{arg}(f''(t_0)) \]

\[ \implies \mathcal{I}(z) \approx \int_C g(t_0) e^{zt_0} e^{\frac{1}{2} |zf''(t_0)| \zeta^2} \, dt \]

or

\[ \mathcal{I}(z) \approx g(t_0) e^{zt_0} e^{\frac{1}{2} |zf''(t_0)| \zeta^2} \int_C \, dt \]

or

\[ \mathcal{I}(z) \approx g(t_0) e^{zt_0} e^{\frac{1}{2} |zf''(t_0)| \zeta^2} \int_C \, dt \]

or

\[ \mathcal{I}(z) \approx g(t_0) e^{zt_0} e^{\frac{1}{2} |zf''(t_0)| \zeta^2} \int_C \, dt \]
where we have chosen \( \alpha \) such that

\[
e^{i(\phi_0 + \phi_{t_0}')} + 2i\alpha = e^{i\pi}
\]

\[
\Rightarrow 2\alpha + \phi_0 + \phi_{t_0}' = \pi
\]

or

\[
\alpha = \frac{\pi - \phi_0 - \phi_{t_0}'}{2}
\]

Because the integrand now decays rapidly we change

\[
\int_0^\infty dt \rightarrow \int_{-\infty}^\infty dt
\]

and use the known integral

\[
\int_{-\infty}^{\infty} e^{-\frac{1}{2} |A| t^2} dt = (\frac{2\pi}{|A|})^{\frac{1}{2}}
\]

to obtain finally

\[
J(t) = q(t_0) e^{i\alpha} e^{\left(\frac{2\pi}{|z + z''(t_0)|}\right)^{\frac{1}{2}}}
\]

\[
\alpha = \pi - \arg(z) - \arg(z''(t_0))
\]

\[
f'(t_0) = 0
\]

Aside: To decide between \( \alpha \) and \( \alpha + \pi \), pick whichever gives an acute angle w.r.t. the original contour. 

New contour
\[ t = z t', \quad dt = z dt \]

\[ \Gamma(z+1) = z \int_0^\infty \frac{e^{z(lnt-t)}}{t} \, dt \]

so \( f(t) = lnt - t, \quad f'(t_0) = \frac{1}{t_0} - 1 = 0 \)

\[ \Rightarrow t_0 = 1, \quad f''(t_0) = -\frac{1}{t_0^2} = -1, \quad f'(t_0) = -1 \]

whereby \( \phi_{f''} = \arg(f''(t_0)) = \pi \)

\[ \Rightarrow \alpha = \pi - \phi_2 - \phi_{f''} = 0 \]

\[ \Rightarrow \Gamma(z+1) \approx e^{-z} e^{\frac{2\pi i}{2}} e^{\left(\frac{2\pi}{1 - (-1)}\right)^{\frac{1}{2}} z^{\frac{1}{2}}} z + 1 \]

or \[ \Gamma(z+1) \approx (2\pi)^{\frac{1}{2}} z^{\frac{1}{2} + \frac{1}{2}} e^{-z} \quad \text{(Stirling's formula)} \]

Some values

<table>
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<tr>
<th>( z )</th>
<th>( \Gamma(z+1) )</th>
<th>Stirling Approx</th>
<th>Fractional error</th>
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<tr>
<td>1</td>
<td>1</td>
<td>0.922</td>
<td>-8%</td>
</tr>
<tr>
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<td>120</td>
<td>118.0</td>
<td>-1.7%</td>
</tr>
<tr>
<td>10</td>
<td>3.63 \times 10^6</td>
<td>3.60 \times 10^6</td>
<td>-0.8%</td>
</tr>
<tr>
<td>15</td>
<td>1.31 \times 10^{12}</td>
<td>1.30 \times 10^{12}</td>
<td>-0.6%</td>
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</table>
Next, suppose \( z = |z| e^{i\phi_z} \) with \( \phi_z \neq 0 \).

Now \( \alpha = \frac{\pi - \phi_z}{2} - \frac{\pi}{2} = -\frac{\phi_z}{2} \).

\[
\lim_{n \to 1} \alpha = -\frac{\phi_z}{2}
\]

\[
to = 1
\]

and our asymptotic approximation gives

\[
\Pi(z+1) = e^{-z} e^{\frac{i\phi_z}{2}} \frac{1}{z^{\frac{1}{2}}} (2\pi)^{\frac{1}{2}} \frac{z}{z^2}
\]

But \( |z|^{\frac{1}{2}} e^{-\frac{i\phi_z}{2}} = \frac{z}{z^2} \).

So for complex \( z \), we still have the same formula that we derived for real, positive \( z \!:

\[
\Pi(z+1) = (2\pi)^{\frac{1}{2}} \frac{z^{\frac{1}{2}}}{z^2} e^{-z}
\]

<table>
<thead>
<tr>
<th>( z )</th>
<th>( \Pi(z+1) )</th>
<th>Stirling Approx</th>
<th>Fractional error</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.509 - 0.111i</td>
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<tr>
<td>( 5i )</td>
<td>-0.0170 - 0.00136i</td>
<td>-0.0168 - 0.00139i</td>
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<tr>
<td>( 10i )</td>
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<td>( 15i )</td>
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