ASTR 5770 Cosmology Fall 2025. Problem Set 6. Due Wed Oct 15

1. Equation of motion (8 points)

The Euler-Lagrange equations of motion for a particle whose Lagrangian is $L(x^\mu, dx^\mu/d\lambda)$ are (this is equation (4.5) of the book),

$$\frac{d}{d\lambda}\frac{\partial L}{\partial(dx^\kappa/d\lambda)} = \frac{\partial L}{\partial x^\kappa} \ . \tag{1.1}$$

In equation (1.1) the coordinates x^μ and velocities $dx^\mu/d\lambda$ in the Lagrangian are to be interpreted as independent quantities, so the velocity partial derivatives $\partial L/\partial(dx^\kappa/d\lambda)$ are done with the coordinates x^μ held fixed, and conversely the coordinate partial derivatives $\partial L/\partial x^\kappa$ are done with the velocities dx^μ/λ held fixed. In general relativity, the effective Lagrangian of a free particle of mass m can be taken to be (this is eq. (4.25) of the book)

$$L = \tfrac{1}{2}\, g_{\mu\nu}(x)\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \tfrac{1}{2}\, g_{\mu\nu}(x) p^\mu p^\nu \ , \tag{1.2}$$

where p^μ is the particle's 4-momentum,

$$p^\mu \equiv \frac{dx^\mu}{d\lambda} = m\frac{dx^\mu}{d\tau} \ , \tag{1.3}$$

with λ the affine parameter along the path of the particle, related to the particle's proper time τ by

$$d\lambda = \frac{d\tau}{m} \ . \tag{1.4}$$

The metric $g_{\mu\nu}$ in the free-particle Lagrangian (1.2) is a function only of the coordinates x^μ, not the velocities p^μ. Show that Lagrange's equations (1.1) are equivalent to the usual equations of motion

$$\frac{d\boldsymbol{p}}{d\lambda} = 0 \ . \tag{1.5}$$

[Hint: First argue that $\boldsymbol{e}^\kappa \cdot \boldsymbol{e}_\mu = \delta^\kappa_\mu$. Then from $\partial(\boldsymbol{e}^\kappa \cdot \boldsymbol{e}_\mu)/\partial x^\nu = 0$ and from the definition of the connection coefficients, $\partial \boldsymbol{e}_\mu/\partial x^\nu \equiv \Gamma^\kappa_{\mu\nu}\boldsymbol{e}_\kappa$, deduce that $\partial \boldsymbol{e}^\kappa/\partial x^\nu = -\Gamma^\kappa_{\mu\nu}\boldsymbol{e}^\mu$. Show that the equation of motion (1.5), with $\boldsymbol{p} = \boldsymbol{e}^\kappa p_\kappa$, becomes $\boldsymbol{e}^\kappa\left(dp_\kappa/d\lambda - \Gamma_{\mu\kappa\nu}p^\mu p^\nu\right) = 0$, where $\Gamma_{\mu\kappa\nu} \equiv g_{\mu\lambda}\Gamma^\lambda_{\kappa\nu}$. Now argue that $\partial L/\partial p^\kappa = p_\kappa$, and that $\partial L/\partial x^\kappa = \Gamma_{\mu\nu\kappa}p^\mu p^\nu$. To complete, you'll need to invoke the no-torsion symmetry $\Gamma_{\mu\nu\kappa} = \Gamma_{\mu\kappa\nu}$.]

2. Geodesics in the FLRW geometry (12 points)

This is Exercise 10.5. The Friedmann-Lemaître-Robertson-Walker metric of cosmology is

$$ds^2 = -\,dt^2 + a(t)^2\left[dx_\parallel^2 + \frac{\sin^2(\kappa^{1/2}x_\parallel)}{\kappa}\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right] \tag{2.1}$$

where κ is a constant, the curvature constant. Note that equation (2.1) is valid for all values of κ, including zero and negative values: there is no need to consider the cases separately.

(a) Conservation of generalized momentum

Consider a particle moving with comoving 4-momentum $p^\mu \equiv dx^\mu/d\lambda$ along a geodesic in the radial direction, so that $d\theta = d\phi = 0$. Argue that the Lagrangian equations of motion

$$\frac{d}{d\lambda}\frac{\partial L}{\partial p^{x_\parallel}} = \frac{\partial L}{\partial x_\parallel} \tag{2.2}$$

with effective Lagrangian

$$L = \tfrac{1}{2}\, g_{\mu\nu} p^\mu p^\nu \tag{2.3}$$

imply that

$$p_{x_\parallel} = \text{constant} \ . \tag{2.4}$$

Argue further from the same Lagrangian equations of motion that the assumption of a radial geodesic is valid because

$$p_\theta = p_\phi = 0 \tag{2.5}$$

is a consistent solution. [Hint: Some components of the metric $g_{\mu\nu}$ depend on the coordinate $x_\parallel$, but for radial geodesics with $p^\theta = p^\phi = 0$, the possible contributions from derivatives of the metric vanish.]

(b) Proper momentum

Argue that a proper interval of distance measured by comoving observers along the radial geodesic is $a\, dx_\parallel$. Hence show from equation (2.4) that the proper momentum $p^{x_\parallel}_{\text{proper}}$ of the particle relative to comoving observers (who are at rest in the FLRW metric) evolves as

$$p^{x_\parallel}_{\text{proper}} \equiv a\frac{dx_\parallel}{d\lambda} \propto \frac{1}{a} \ . \tag{2.6}$$

(c) Redshift

What relation does your result (2.6) imply between the redshift $1 + z$ of a distant object observed on Earth and the expansion factor a since the object emitted its light?

(d) Temperature of the CMB

Argue from the above results that the temperature T of the CMB evolves with cosmic scale factor as

$$T \propto \frac{1}{a} \ . \tag{2.7}$$