ASTR 5770 Cosmology Fall 2025. Problem Set 5. Due Wed Oct 1

1. Equations of motion in weak gravity (12 points)

This is Exercise 2.17. Consider the Newtonian metric

$$ds^2 = -(1+2\Phi)dt^2 + (1-2\Phi)(dx^2 + dy^2 + dz^2) \ , \tag{1.1}$$

where $\Phi(x, y, z)$ is the familiar Newtonian gravitational potential, a function only of the spatial coordinates x, y, z, not of time t.

(a) Connection coefficients

Confirm that the non-zero connection coefficients are (coefficients as below but with the last two indices swapped are the same by the no-torsion condition $\Gamma^\kappa_{\mu\nu} = \Gamma^\kappa_{\nu\mu}$)

$$\Gamma^t_{t\alpha} = \Gamma^\alpha_{tt} = \Gamma^\alpha_{\beta\beta} = -\Gamma^\beta_{\beta\alpha} = -\Gamma^\alpha_{\alpha\alpha} = \frac{\partial \Phi}{\partial x^\alpha} \quad (\alpha \neq \beta = x, y, z) \ . \tag{1.2}$$

[Hint: Work to linear order in Φ. You are welcome to use the mathematica notebook metric.nb posted on the website, but if you do, please tell me.]

(b) Energy of a massive particle

Consider a massive, non-relativistic particle moving with 4-velocity $u^\mu \equiv dx^\mu/d\tau = \{u^t, u^x, u^y, u^z\}$. Show that $u_\mu u^\mu = -1$ implies that

$$u^t = 1 + \frac{1}{2}u^2 - \Phi \ , \tag{1.3}$$

whereas

$$u_t = -\left(1 + \frac{1}{2}u^2 + \Phi\right) \ , \tag{1.4}$$

where $u \equiv [(u^x)^2 + (u^y)^2 + (u^z)^2]^{1/2}$. One of u^t or u_t is constant along the geodesic. Which one? [Hint: Work to linear order in Φ. Note that u^2 is of linear order in Φ. As regards which of u^t or u_t is constant, notice that the metric is independent of time because $\Phi(x, y, z)$ is being assumed to be a function only of the spatial coordinates x, y, z, not of time t.]

(c) Equation of motion of a massive particle

From the geodesic equation

$$\frac{du^\kappa}{d\tau} + \Gamma^\kappa_{\mu\nu} u^\mu u^\nu = 0 \ , \tag{1.5}$$

show that

$$\frac{du^\alpha}{dt} = -\frac{\partial \Phi}{\partial x^\alpha} \quad \text{for } \alpha = x, y, z \ . \tag{1.6}$$

Why is it legitimate to replace $d\tau$ by dt? Show further that

$$\frac{du^t}{dt} = -2u^\alpha \frac{\partial \Phi}{\partial x^\alpha} \ , \tag{1.7}$$

with implicit summation over $\alpha = x, y, z$. Does the result agree with what you'd expect from equation (1.3)? [Hint: A consistent perturbative approach is to keep only the lowest order non-vanishing parts of an equation, discarding the higher order parts as negligible.]

(d) Energy of a massless particle

For a massless particle, the proper time along a geodesic is zero, and the affine parameter λ must be used instead of the proper time. The 4-velocity of a massless particle can be defined to be (and really this is just the 4-momentum up to an arbitrary overall factor) $v^\mu \equiv dx^\mu/d\lambda = \{v^t, v^x, v^y, v^z\}$. Show that $v_\mu v^\mu = 0$ implies that

$$v^t = (1 - 2\Phi)v \ , \tag{1.8}$$

whereas

$$v_t = -v \ , \tag{1.9}$$

where $v \equiv [(v^x)^2 + (v^y)^2 + (v^z)^2]^{1/2}$. One of v^t or v_t is constant. Which one?

(e) Equation of motion of a massless particle

From the geodesic equation

$$\frac{dv^\kappa}{d\lambda} + \Gamma^\kappa_{\mu\nu} v^\mu v^\nu = 0 \tag{1.10}$$

show that the spatial components $\boldsymbol{v} \equiv \{v^x, v^y, v^z\}$ satisfy

$$\frac{d\boldsymbol{v}}{d\lambda} = 2\,\boldsymbol{v} \times (\boldsymbol{v} \times \boldsymbol{\nabla}\Phi) \tag{1.11}$$

where boldface symbols represent 3D vectors, and in particular $\boldsymbol{\nabla}\Phi$ is the spatial 3D gradient $\boldsymbol{\nabla}\Phi \equiv \partial\Phi/\partial x^\alpha = \{\partial\Phi/\partial x, \partial\Phi/\partial y, \partial\Phi/\partial z\}$. [Hint: Recall that the 3D vector triple product satisfies $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$.]

(f) Interpret

Interpret your answer, equation (1.11). In what ways does this equation for the acceleration of photons differ from the equation governing the acceleration of massive particles? [Hint: Without loss of generality, the affine parameter can be normalized so that the photon speed is one, $v = 1$, so that $\boldsymbol{v}$ is a unit vector representing the direction of the photon.]

2. Deflection of light by the Sun (8 points)

This is Exercise 2.18.

(a) Born approximation

Consider light that passes by a spherical mass M sufficiently far away that the potential Φ is always weak. The potential at distance r from the spherical mass can be approximated by the Newtonian potential

$$\Phi = -\frac{GM}{r} \ . \tag{2.1}$$

Approximate the unperturbed path of light past the mass as a straight line. The plan is to calculate the deflection as a perturbation to the straight line (physicists call this the Born approximation). For definiteness, take the light to be moving in the x-direction, offset by a constant amount y away from the mass in the y-direction (so y is the impact parameter, or periapsis). Argue that equation (1.11) becomes

$$\frac{dv^y}{d\lambda} = v^x \frac{dv^y}{dx} = -2\,(v^x)^2 \frac{\partial \Phi}{\partial y} \ . \tag{2.2}$$

Integrate this equation to show that

$$\frac{\Delta v^y}{v^x} = -\frac{4GM}{y} \ . \tag{2.3}$$

Argue that this equals the deflection angle $\Delta\phi$.

(b) Deflection of light by the Sun

Calculate the predicted deflection angle $\Delta\phi$ in arcseconds for light that just grazes the limb of the Sun.

(c) Compare to observations

A remarkable test of the prediction was accomplished recently from observations of the 21 August 2017 Eclipse by amateur astronomer and CU alumnus Don Bruns (Physics & Mathematics 1974), as reported in Physics Today, August 2018. Briefly, what did Don Bruns find?