ASTR 5720 Galaxies & Cosmology Fall 1997. FAQ for Problem Set 4.

1. Some useful constants

The constants in the following employ Cohen & Taylor's (1986) least-squares adjustment of the physical constants. A web link to these constants is given on the references page linked to the ASTR 5720 home page.

$1\,\mathrm{eV} = 1.60217733 \times 10^{-12}\,\mathrm{erg}$ is equivalent to $11604.45\,\mathrm{K}$.

The Planck energy is

$$(\hbar c^5/G)^{1/2} = 1.95633 \times 10^{16}\,\mathrm{erg} = 1.22104 \times 10^{19}\,\mathrm{GeV}\ . \tag{1.1}$$

The electron mass is (the final expression below is of course really $m_e c^2$)

$$m_e = 9.1093897 \times 10^{-28}\mathrm{gm} = 0.51099906\,\mathrm{MeV}\ . \tag{1.2}$$

The hydrogen ionization energy is

$$\chi_{\mathrm{H}} = 13.5984379\,\mathrm{eV} \tag{1.3}$$

which differs slightly from the Rydberg, which is half an atomic mass unit, $m_e c^2 \alpha^2/2 = 13.6056981\,\mathrm{eV}$ with $\alpha = 1/137.0359895$ the fine-structure constant.

The CMB temperature today is (Fixsen et al 1996 ApJ 473, 576)

$$T_0 = 2.728 \pm 0.004\,\mathrm{K} = 2.351 \pm 0.004 \times 10^{-4}\,\mathrm{eV} \quad (95\%\ \text{confidence})\ . \tag{1.4}$$

The corresponding neutrino temperature, for neutrinos which remain relativistic today, is

$$T_{\nu,0} = \left(\frac{4}{11}\right)^{1/3} T_0 = 1.947 \pm 0.003\,\mathrm{K} = 1.678 \pm 0.003 \times 10^{-4}\,\mathrm{eV}\ . \tag{1.5}$$

I tried searching on the web for the IAU (International Astronomical Union) definitive list of astronomical constants, but they don't seem to have such a page on the web. I got the following from Eric's Treasure Troves of Science, which is linked on the reference page on the ASTR 5720 pages. One Astronomical Unit, the Earth-Sun distance, was defined by the IAU in 1976 to be

$$1\,\mathrm{AU} \equiv 149,597,870\,\mathrm{km}\ . \tag{1.6}$$

The parsec is the distance at which 1 AU subtends one arcsecond, which is

$$1\,\mathrm{pc} \equiv \frac{648,000}{\pi}\,\mathrm{AU} = 3.08567757 \times 10^{13}\,\mathrm{km}\ . \tag{1.7}$$

2. What is Omega in photons today?

The present day ratio $\Omega_{\gamma,0}$ of the mass-energy density $\rho_{\gamma,0}$ of CMB photons to the critical density ρ_c is

$$\Omega_{\gamma,0} \equiv \frac{\rho_{\gamma,0}}{\rho_c} = \frac{8\pi G\rho_{\gamma,0}}{3H_0^2} = \frac{8\pi^3 G(kT_0)^4}{45H_0^2 c^5 \hbar^3} = 2.48\times 10^{-5}\, h^{-2}\, T_{2.728\mathrm{K}}^4 \tag{2.1}$$

where $h \equiv H_0/(100\,\mathrm{km\,s^{-1}\,Mpc^{-1}})$, and $T_{2.728\mathrm{K}} \equiv T_0/2.728\,\mathrm{K}$.

3. What is the relation between Ω_ν and m_ν?

The number density of each of the three light neutrino types (particle and anti-particle) to CMB photons since electon-positron annihilation has been

$$\frac{n_\nu}{n_\gamma} = \frac{3}{4}\left(\frac{T_\nu}{T}\right)^3 = \frac{3}{11}\ . \tag{3.1}$$

For a neutrino type mass m_ν massive enough to be nonrelativistic today, the ratio $\Omega_{\nu,0}$ of its mass density $\rho_{\nu,0}$ to the critical density ρ_c is

$$\Omega_{\nu,0} \equiv \frac{\rho_{\nu,0}}{\rho_c} = \frac{\Omega_{\gamma,0} m_\nu n_{\nu,0}}{\rho_{\gamma,0}} = \frac{\Omega_{\gamma,0} m_\nu \frac{3}{11} n_{\gamma,0}}{\rho_{\gamma,0}} = \frac{3\,\Omega_{\gamma,0}}{11\,\epsilon^-}\frac{m_\nu c^2}{kT_0} = \left(\frac{m_\nu c^2}{93.9\,\mathrm{eV}}\right) h^{-2} T_{2.728\mathrm{K}}^3 \tag{3.2}$$

where ϵ^- is the mean energy per photon in units of kT

$$\epsilon^- \equiv \frac{\rho_\gamma c^2}{n_\gamma kT} = \frac{F_4^-}{F_3^-} = \frac{\pi^4}{30\zeta(3)} = 2.7011780 \tag{3.3}$$

with $\zeta(3) = 1.2020569$ a Riemann zeta-function.

4. What is the relation between the baryon-to-photon ratio η and Ω_b?

Baryons in the Universe today are almost all protons and neutrons, combined into nuclei of one sort another. The term 'baryonic matter' in cosmology is generally used to mean normal matter, including electrons (which are really leptons). The parameter η is by definition the present day baryon-to-photon ratio, something which is not presently, but which one day may be, calculable from first principles,

$$\eta \equiv \frac{n_{b,0}}{n_{\gamma,0}}\ . \tag{4.1}$$

The ratio $\Omega_{b,0}$ of baryonic matter density to the critical density is related to η by

$$\Omega_{b,0} \equiv \frac{\rho_{b,0}}{\rho_c} = \frac{\Omega_{\gamma,0} m_b n_{b,0}}{\rho_{\gamma 0}} = \frac{\eta\Omega_{\gamma,0}}{\epsilon^-}\frac{m_b c^2}{kT_0} = 3.67\times 10^7\,\eta\, h^{-2} T_{2.728\mathrm{K}}^3 \tag{4.2}$$

where $m_b = 939\,\mathrm{MeV}$ is the approximate mass per baryon.

5. What was the Hubble constant in the vacuum-dominated regime?

By definition, the Hubble constant H is related to the critical density ρ_c at any epoch by

$$H^2 = \frac{8\pi G\rho_c}{3} \ . \tag{5.1}$$

In the vacuum-dominated phase, the vacuum mass-energy density ρ_Λ (Einstein's cosmological constant Λ is related to vacuum energy by $\Lambda = 8\pi G\rho_\Lambda$) can be expressed in terms of a mass scale m_Λ as

$$\rho_\Lambda = \frac{m_\Lambda^4 c^3}{\hbar^3} \ . \tag{5.2}$$

Inflation quickly drives Ω to one, so $\rho_c = \rho_\Lambda$. The Hubble constant in the vacuum-dominated phase is then

$$H_\Lambda = \left(\frac{8\pi G m_\Lambda^4 c^3}{3\,\hbar^3}\right)^{1/2} \ . \tag{5.3}$$

The ratio of H_Λ to the present day Hubble constant H_0 can be written in the following useful fashion

$$\frac{H_\Lambda}{H_0} = \left(\frac{\Omega_{\gamma,0}\,\rho_\Lambda}{\rho_{\gamma,0}}\right)^{1/2} = \left(\frac{15\,\Omega_{\gamma,0}}{\pi^2}\right)^{1/2}\left(\frac{m_\Lambda c^2}{kT_0}\right)^2 = 1.66\times 10^{61}\,h^{-1}\left(\frac{m_\Lambda}{m_P}\right)^2 \tag{5.4}$$

where m_P is the Planck mass.

6. What was the Hubble constant in the radiation-dominated regime?

Radiation-dominated means the mass-energy density of the Universe was dominated by relativistic particles. Radiation here includes not just photons, but any other particles which happened to be relativistic. In the radiation-dominated regime, Ω was equal to one to an excellent approximation, and the critical density was just equal to the density in relativistic particles, $\rho_c = \rho_{\rm rel}$, which was

$$\rho_{\rm rel} = \frac{\pi^2 (kT)^4}{30\,c^5\hbar^3}\,g^* \tag{6.1}$$

where g^* was the effective number of relativistic particle types,

$$g^* = g_B + \frac{7}{8}\,g_F \ . \tag{6.2}$$

Particle accelerators currently probe particle energies up to $\sim 1\,\mathrm{TeV}$, so g^* is not known with any confidence above these energies. The convergence of the 3 renormalized coupling constants at $\sim 10^{15}$ GeV in the standard $SU(3)\times SU(2)\times U(1)_Y$ theory suggests a possible Grand Unification of the color and electroweak forces at this GUT energy $\sim 10^{15}$ GeV. If nothing else happens (the GUT desert), then at temperatures $T \gtrsim 300\,\mathrm{GeV}$ up to the GUT temperature the Universe is quark soup, and

$$g^* = 28 + \frac{7}{8}\,90 = 106.75 \quad \text{(quark soup)} \tag{6.3}$$

composed of: 14 boson types — 8 $SU(3)$ color gluons, the 3 $SU(2)$ $W^{\pm,0}$ and 1 $U(1)_Y$ B electroweak bosons (at low energy the neutral W^0 and B twist into the Z^0 and the photon γ), and 1 complex Higgs doublet — together with their CP conjugates, i.e. the anti-particle of opposite spin; and 3 generations of 15 fermion types — 3 colors (rgb) of 2 flavors of quark (the down d and up u in the 1st generation, the strange s and charm c in the 2nd, the bottom b and top t in the 3rd), a lepton (the electron e in the 1st generation, the muon μ in the 2nd, the tauon τ in the 3rd), each of those quarks and leptons coming in left- and right-handed versions, and finally a neutrino (ν_e, ν_μ, ν_τ), which comes only in a left-handed version — again together with their CP conjugates.

It's a good bet that the GUT desert is a mirage. In the standard model, the three coupling constants (α_3 for $SU(3)$, α_2 for $SU(2)$, α_1 for $U(1)_Y$) don't quite converge at the GUT energy. The coupling constants can be made to converge nicely however by incorporating supersymmetry, spontaneously broken at the $\sim 1\,\mathrm{TeV}$ scale. If this is true, then g^* would be larger than the quark soup value (6.3) at energies above the supersymmetry breaking scale.

At temperatures $100\,\mathrm{MeV} \lesssim T \lesssim 1\,\mathrm{MeV}$ when photons, neutrinos, electrons and positrons (and no other particles) were relativistic, the effective particle number g^* was

$$g^* = 2 + \frac{7}{8} 10 = 10.75 \quad (\gamma,\, \nu,\, e)\ . \tag{6.4}$$

Following electron-positron annihilation, at $T \lesssim 0.1\,\mathrm{MeV}$, the neutrino-to-photon temperature was $T_\nu/T = (4/11)^{1/3} = 0.714$, and the effective particle number g^* was

$$g^* = 2 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} 6 = 3.36 \quad (\gamma,\, 3 \text{ relativistic } \nu \text{ types})\ . \tag{6.5}$$

If one of the three neutrino types is massive, it can happen that only two of the three neutrino types is relativistic at the time the Universe goes from radiation- to matter-dominated. In that case, the effective particle number g^* would have been

$$g^* = 2 + \frac{7}{8}\left(\frac{4}{11}\right)^{4/3} 4 = 2.91 \quad (\gamma,\, 2 \text{ relativistic } \nu \text{ types})\ . \tag{6.6}$$

For values of g^* at intermediate times, see Kolb & Turner (1990) "The Early Universe", Fig. 3.5.

The ratio of the Hubble constant H in the radiation-dominated regime at photon temperature T, to the Hubble constant H_0 today can be written in the following useful form

$$\frac{H}{H_0} = \left(\frac{\Omega_{\gamma,0}\,\rho_{\mathrm{rel}}}{\rho_{\gamma,0}}\right)^{1/2} = \left(\frac{\Omega_{\gamma,0}\,g^*}{2}\right)^{1/2}\left(\frac{T}{T_0}\right)^2 = \left(\frac{2.48\times 10^{-5}\,g^*}{2}\right)^{1/2}\left(\frac{T}{2.728\,\mathrm{K}}\right)^2 h^{-1}\ . \tag{6.7}$$

7. What was the Hubble constant in the matter-dominated regime?

In the matter-dominated regime, the ratio of Ω at redshift z to Ω_0 at the present time is

$$\frac{\Omega}{\Omega_0} = \left(\frac{1+\Omega_0 z}{1+z}\right)^{1/2} \tag{7.1}$$

where $1 + z = a_0/a$. In the matter-dominated regime, the ratio of the Hubble constant H at redshift z to the Hubble constant H_0 today is

$$\frac{H}{H_0} = \left(\frac{\Omega_0 \rho}{\Omega \rho_0}\right)^{1/2} = (1 + \Omega_0 z)^{1/2}(1 + z) \ . \tag{7.2}$$

8. What was the cosmic scale factor since reheating?

The dimensionless entropy S/k in a volume V of the Universe is to a good approximation given by the entropy in relativistic particles, which is

$$\frac{S}{k} = \frac{2\pi^2 (kT)^3 V}{45 c^3 \hbar^3}\, g_s^* \tag{8.1}$$

where g_s^* is an effective number of relativistic particle types. Before electron-positron annihilation, this effective particle number was the same as g^*,

$$g_s^* = g^* \quad \text{(before } e\text{-}\bar{e} \text{ annihilation)} \ . \tag{8.2}$$

After electron-positron annihilation, the effective particle number was

$$g_s^* = 2 + \frac{7}{8}\frac{4}{11} 6 = \frac{43}{11} = 3.91 \quad \text{(after } e\text{-}\bar{e} \text{ annihilation)} \tag{8.3}$$

which remains the value of g_s^* today, entropy having being conserved separately in photons and each of the three neutrino types.

Conservation of entropy in comoving volumes implies

$$\frac{a_0}{a} = \left(\frac{g_s^*}{g_{s,0}^*}\right)^{1/3} \frac{T}{T_0} = \left(\frac{g_s^*}{43/11}\right)^{1/3} \frac{T}{2.728\,\mathrm{K}} \tag{8.4}$$

where T was the photon temperature. Equation (8.4) has been valid since the time of reheating at the end of inflation, when most of the entropy of the Universe (apart from that in black holes) today was created. It is valid through the radiation-dominated regime, up to the present time, and is valid irrespective of whether or not there is a cosmological constant today.