PHYS 7810 Extreme Physics Spring 2026. Problem Set 2. Due Feb 5

1. The mechanism of mass inflation — 20 points

This is essentially Exercise 21.1 of the book. Einstein's equations in a spherically symmetric spacetime imply that the covariant rate of change of the radial 4-gradient $\beta_m \equiv \partial_m r = \{\partial_0 r, \partial_1 r, 0, 0\}$ in the frame of any radially moving orthonormal tetrad is

$$D_0\beta_0 = -\frac{M}{r^2} - 4\pi r p \ , \tag{1.1a}$$

$$D_0\beta_1 = 4\pi r f \ , \tag{1.1b}$$

where D_0 is the covariant time derivative, p is the radial pressure, f is the radial energy flux, and M is the interior mass defined by

$$\frac{2M}{r} - 1 \equiv \beta^2 \equiv -\beta_m\beta^m = \beta_0^2 - \beta_1{}^2 \ . \tag{1.2}$$

(a) Freely-falling stream (4 points)

Consider a stream of matter that is freely falling radially inside the horizon of a spherically symmetric black hole. Let u be the radial component of the 4-velocity of the stream relative to the "no-going" frame where $\beta_1 = 0$ (the frame of reference that divides outgoing frames $\beta_1 < 0$ from ingoing frames $\beta_1 > 0$)

$$u^m \equiv \{-\beta_0/\beta, -\beta_1/\beta, 0, 0\} \equiv \{\sqrt{1+u^2}, u, 0, 0\} \ . \tag{1.3}$$

Note that β_0 is negative inside the horizon. The time component $u^t \equiv -\beta_0/\beta = \sqrt{1+u^2}$ of the 4-velocity is positive (as it should be for a proper 4-velocity), while the radial component $u \equiv u^1 \equiv -\beta_1/\beta$ of the 4-velocity is positive outgoing, negative ingoing. Show that along the worldline of the stream,

$$\frac{d\ln\beta}{d\ln r} = \frac{1}{\beta^2}\left[-\frac{M}{r} - 4\pi r^2\left(p + \frac{\beta_1}{\beta_0} f\right)\right] \ , \tag{1.4a}$$

$$\frac{d\ln u}{d\ln r} = \frac{1}{\beta^2}\left[\frac{M}{r} + 4\pi r^2\left(p + \frac{\beta_0}{\beta_1} f\right)\right] \ . \tag{1.4b}$$

[Hint: If the stream is freely falling, then the proper time derivative ∂_0 in the tetrad frame of the stream equals the covariant time derivative D_0. Thus the proper rates of change of $\ln\beta$ and $\ln u$ with respect to $\ln r$ along the worldline of the stream are

$$\frac{d\ln\beta}{d\ln r} = \frac{\partial_0 \ln\beta}{\partial_0 \ln r} \ , \quad \frac{d\ln u}{d\ln r} = \frac{\partial_0 \ln u}{\partial_0 \ln r} \ . \tag{1.5}$$

These can be evaluated through

$$\partial_0 \ln\beta = D_0 \ln\beta = \frac{1}{2\beta^2} D_0\beta^2 = \frac{1}{2\beta^2} D_0(\beta_0^2 - \beta_1^2) = \frac{1}{\beta^2}(\beta_0 D_0\beta_0 - \beta_1 D_0\beta_1) \ , \tag{1.6a}$$

$$\partial_0 \ln u = D_0 \ln u = D_0 \ln\beta_1 - D_0 \ln\beta = \frac{1}{\beta_1} D_0\beta_1 - D_0\ln\beta \ , \tag{1.6b}$$

$$\partial_0 \ln r = \frac{1}{r}\,\partial_0 r = \frac{\beta_0}{r} \ , \tag{1.6c}$$

with Einstein's equations (1.1) substituted into equations (1.6a) and (1.6b).]

(b) Equal outgoing and ingoing streams (4 points)

Consider the symmetrical case of two equal streams of radially outgoing ($\beta_1 < 0$) and ingoing ($\beta_1 > 0$) neutral, pressureless, non-interacting matter ("dust"), each of proper density ρ in their own frames, freely falling into a charged black hole. Show that

$$\frac{d\ln\beta}{d\ln r} = -\frac{1}{2\beta^2}\left(-\lambda + \beta^2 + \mu u^2\right) \ , \tag{1.7a}$$

$$\frac{d\ln u}{d\ln r} = -\frac{1}{2\beta^2}\left(\lambda - \beta^2 + \mu + \mu u^2\right) \ , \tag{1.7b}$$

where

$$\lambda \equiv Q^2/r^2 - 1 \ , \quad \mu \equiv 16\pi r^2 \rho \ . \tag{1.8}$$

Hence conclude that

$$\frac{d\ln\beta}{d\ln u} = \frac{-\lambda + \beta^2 + \mu u^2}{\lambda - \beta^2 + \mu + \mu u^2} \ . \tag{1.9}$$

[Hint: The assumption that the streams are neutral, pressureless, and non-interacting is needed to make the streams freely falling, so that equations (1.4) are valid. The pressure p in the tetrad frame of each stream is the sum of the electromagnetic pressure p_e and the streaming pressure p_s

$$p = p_e + p_s \ . \tag{1.10}$$

The electromagnetic pressure p_e is

$$p_e = -\frac{Q^2}{8\pi r^4} \ , \tag{1.11}$$

with Q the charge of the black hole, which is constant because the infalling streams are neutral. The streaming pressure p_s that each stream sees is

$$p_s = \rho (u_s^1)^2 \ , \tag{1.12}$$

where the streaming 4-velocity u_s^m between the two streams is the 4-velocity of the observed stream Lorentz-boosted by the 4-velocity of the observing stream (the radial velocities u^1 of the observed and observing streams have opposite signs)

$$u_s^0 = (u^0)^2 + (u^1)^2 = 1 + 2u^2 \ , \quad u_s^1 = -2u^0 u^1 = -2u\sqrt{1+u^2} \ . \tag{1.13}$$

The energy flux f in the tetrad frame of each stream is the streaming flux f_s

$$f = f_s = \rho u_s^0 u_s^1 \ . \tag{1.14}$$

You should find that the combinations of streaming pressure and flux that go into equations (1.4) are

$$p_s + \frac{\beta_1}{\beta_0} f_s = 2\rho u^2 \ , \tag{1.15a}$$

$$p_s + \frac{\beta_0}{\beta_1} f_s = -2\rho(1 + u^2) \ . \tag{1.15b}$$

(c) Reissner-Nordström phase (3 points)

If the accretion rate is small, then initially the stream density ρ is small, and consequently μ is small. Argue that in this regime equation (1.9) simplifies to

$$\frac{d\ln\beta}{d\ln u} = \frac{-\lambda + \beta^2}{\lambda - \beta^2} \ . \tag{1.16}$$

Hence conclude that

$$\beta = \frac{C}{u} \ , \tag{1.17}$$

where C is some constant set by initial conditions (generically, C will be of order unity).

(d) Transition to mass inflation (3 points)

Argue that in the Reissner-Nordström phase, β becomes small, and u grows large, as the streams fall to smaller radius r. Argue that in due course equation (1.9) becomes well-approximated by

$$\frac{d\ln\beta}{d\ln u} = \frac{-\lambda + \mu u^2}{\lambda + \mu u^2} \ . \tag{1.18}$$

Treating λ and μ as constants (which is a good approximation), show that the solution to equation (1.18) subject to the initial condition set by equation (1.17) is

$$\beta = \frac{C(\lambda + \mu u^2)}{\lambda u} \ . \tag{1.19}$$

[Hint: λ is positive. In the Reissner-Nordström solution, β would go to zero at the inner horizon.]

(e) Sketch (3 points)

Sketch the solution (1.19), plotting u against β on logarithmic axes. Mark the regime where mass inflation is occurring.

(f) Inflationary growth rate (3 points)

Argue that during mass inflation the inflationary growth rate $d\ln\beta/d\ln r$ is

$$\frac{d\ln\beta}{d\ln r} = -\frac{\lambda^2}{2C^2\mu} \ . \tag{1.20}$$

Comment on how the inflationary growth rate depends on accretion rate (on ρ).