Wronskian

Q: When are N functions $y_i(x)$ linearly dependent?

A: If $\exists$ $\overset{\text{constants}}{a_i}$ s.t. $\sum_{i=1}^{N} a_i y_i(x) = 0$

Spose $\sum_{i=1}^{N} a_i y_i = 0$

then

$$\left.\begin{aligned} \sum a_i y_i' &= 0 \\ \sum a_i y_i'' &= 0 \\ &\vdots \\ \sum a_i y_i^{(N-1)} &= 0 \end{aligned}\right\} \text{ Set of N equations.}$$

is set of linear equations, which can can be written in matrix form

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_N \\ y_1' & y_2' & \cdots & y_N'' \\ & & & \\ y_1^{(N-1)} & y_2^{(N-1)} & \cdots & y_N^{(N-1)} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

This has a solution
(in which a_i is an eigenvector with 0 eigval)
iff

$$\underbrace{\begin{vmatrix} y_1 & y_2 & \cdots & y_N \\ y_1' & y_2' & \cdots & y_N' \\ & & & \\ y_1^{(N-1)} & & \cdots & y_N^{(N-1)} \end{vmatrix}}_{\text{is called } \underline{\text{Wronskian}}} = 0$$

A: $y_i(x)$ are lin. dept iff Wronskian $= 0$ at all x.

Wronskian method to find
2nd solution to 2nd order lin homog eqn

$$y'' + p(x)\, y' + q(x)\, y = 0$$

has 2 solutions $y_1(x)$, $y_2(x)$.
Their Wronskian is

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Differentiate W:

$$W' = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'}$$

$$= y_1 y_2'' - y_1'' y_2$$

$$\underset{\text{diffeq}}{\overset{\text{use}}{=}} y_1(-p y_2' - q y_2) - y_2(-p y_1' - q y_1)$$

$$= -p\,(y_1 y_2' - y_1' y_2)$$

$$= -pW$$

ie $\dfrac{d \ln W}{dx} = -p$

which has soln $\ln W = -\int p(x)\, dx + \text{const}$

ie $\boxed{W = \text{const.}\; e^{-\int p(x)\, dx}}$

So if you know one solution y_1
then $y_1 y_2' - y_1' y_2 = W$

ie $y_1^2 \dfrac{d}{dx}\left(\dfrac{y_2}{y_1}\right) = W$

ie $\boxed{y_2 = y_1 \left[\int \dfrac{W}{y_1^2}\, dx + \text{const} \right]}$

gives 1st soln, so set to zero wlog.

<u>N'th order Wronskian</u> of $y^{(N)} + \sum_{\alpha=0}^{(N-1)} p_\alpha y^{(\alpha)} = 0$

$$W = \sum_{ij\cdots m} \varepsilon_{ij\cdots m}\, y_i y_j' \cdots y_m^{(N-1)}$$

$$W' = \sum \varepsilon_{ij\cdots m} \Big(\underbrace{y_i' y_j'}_{\text{cancel by antisymmetry}} \cdots y_m^{(N-1)} + \cdots + \underbrace{y_i y_j' \cdots y_m^{(N)}}_{\substack{\text{only one}\\ \text{that does not}\\ \text{cancel by antisym}}} \Big)$$

$$= -\sum \varepsilon_{ij\cdots m}\, y_i y_j' \cdots \sum \underbrace{p_0 y_m + p_1 y_m^{(1)} +}_{\substack{\text{all cancel by}\\ \text{antisymmetry}}} \cdots + \underbrace{p_{N-1} y_m^{(N-1)}}_{\substack{\text{except}\\ \text{this one}}}$$

$$= -p_{N-1} \sum \varepsilon_{ij\cdots m}\, y_i y_j' \cdots y_m^{(N-1)}$$

$$= -p_{N-1} W .$$

ie

$$\boxed{\frac{d\ln W}{dx} = -p_{N-1}} \quad \text{as before}$$

Spose already know $N-1$ solns $y_1 \cdots y_{N-1}$.
Then, with soln for Wronskian in hand,

$$\begin{vmatrix} y_1 & \cdots & y_N \\ y_1' & & \\ y_1^{(N-1)} & \cdots & y_N^{(N-1)} \end{vmatrix} = W$$

is $(N-1)$'th order linear diff eq for y_N.
In other words system of diff eqs has been reduced by one order.

Ex/

$$y'' + k^2 y = 0$$

Take $y_1 = \sin kx$

Here $p(x) = 0$

So $\ln W = \text{constant}$

Take $W = 1$ wlog

$$y_2 = y_1 \left[\int \frac{W}{y_1^2}\,dx + \cancel{\text{constant}} \right]$$

0 wlog as it contributes to 1st solution

$$= \sin kx \int \frac{W}{\sin^2 kx}\,dx$$

$$= \sin kx \cdot \frac{-W \cos kx}{k \sin kx}$$

$$= \underbrace{-\frac{W}{k}}_{\text{constant}} \cos kx$$

$\Rightarrow$ 2nd solution is $\cos kx$. ✓

Case $a^2 = b$ in $y'' + 2ay' + by = 0$

$$y'' + 2ay' + a^2 y = 0$$

Trial $y = e^{\lambda x}$ yields $(\lambda^2 + 2a\lambda + a^2)y = 0$

ie $(\lambda + a)^2 y = 0$

only one soln

set = 1 wlog

$$y_1 = \cancel{\text{const}} \,.\, e^{-ax}$$

1st soln

set = 1 wlog

Here $p(x) = 2a$

so Wronskian $W = \cancel{\text{const}} \,.\, e^{-\int p(x)\,dx}$

$$= e^{-2ax}$$

Hence

$$\underset{\text{2nd soln}}{y_2} = \cancel{y_1} \int \frac{W}{y_1^2}\, dx$$

e^{-ax} e^{-2ax} e^{-2ax}

$$= e^{-ax} \int dx$$

ie 2nd soln is

$$y_2 = e^{-ax}\, x$$

RHB Ch.16

Frobenius' Method = Series Solution of ODE

Take for example 2nd order linear homog
(but none of those was necessary)

$$y'' + p(x)\,y' + q(x)\,y = 0$$

Try
$$y = \sum_n a_n x^n$$
$$y' = \sum_n n a_n x^{n-1}$$
$$y'' = \sum_n (n-1) n a_n x^{n-2}$$

range of n unspecified, the eqs themselves will tell you what range to use

So

$$\underbrace{\sum_n (n-1) n a_n x^{n-2}}_{n-2=m} + \underbrace{\overbrace{\sum_k p_k x^k}^{\text{Taylor expansion of } p} \sum_n n a_n x^{n-1}}_{k+n-1=m} + \underbrace{\overbrace{\sum_k q_k x^k}^{\text{Taylor expansion of } q} \sum_n a_n x^n}_{k+n=m} = 0$$

$$= \sum_m x^m \underbrace{\left[(m+1)(m+2) a_{m+2} + \sum_n p_{m+1-n}\, n a_n + \sum_n q_{m-n}\, a_n \right]}_{=0\ \forall m} = 0$$

$= 0 \ \forall x$ requires coeffs $= 0 \ \forall m$

Ex/ Simple harmonic oscillator.

$$y'' + k^2 y = 0$$

$$\sum (n-1) n a_n x^{n-2} + k^2 \sum a_n x^n = 0$$

ie $$\sum \left[(n+1)(n+2) a_{n+2} + k^2 a_n \right] x^n = 0$$

$$\Rightarrow (n+1)(n+2) a_{n+2} + k^2 a_n = 0$$

"indicial eq"

ie $$a_{n+2} = \frac{-k^2 a_n}{(n+1)(n+2)} \quad \text{unless } (n+1)(n+2) = 0$$

Starting pt of series set by $(n+1)(n+2) a_{n+2} = 0$.

Yields two solutions,

one beginning with a_0, the other with a_1

$$a_0 \left(1 - \frac{k^2 x^2}{2!} + \frac{k^4 x^4}{4!} - \dots \right) = a_0 \cos kx$$

$$\frac{a_1}{k} \left(kx - \frac{k^3 x^3}{3!} + \frac{k^5 x^5}{5!} - \dots \right) = \frac{a_1}{k} \sin kx$$

Fuch's Theorem

A 2nd order linear ODE has at least one series solution, about $x = x_0$ say, provided that

(a) p, q analytic at $x = x_0$ ' ordinary pt

or (b) $(x-x_0)p$ and $(x-x_0)^2 p$ analytic at $x = x_0$

regular singular pt.

Problem: 2 power series solutions don't always exist.

Ex/ Bessel equation

$$x^2 y'' + x y' + (x^2 - \ell^2) y = 0$$

$$\sum (n-1) n a_n x^n + \sum n a_n x^n + \sum a_n x^{n+2} - \ell^2 \sum a_n x^n = 0$$

$$= \sum \left[(n-1) n a_n + n a_n + a_{n-2} - \ell^2 a_n \right] x^n = 0$$

$$\Rightarrow \quad (n^2 - \ell^2) a_n + a_{n-2} = 0$$

$$\Rightarrow \quad a_n = \frac{-a_{n-2}}{(n-\ell)(n+\ell)} \qquad \text{unless } \overbrace{n^2 - \ell^2 = 0}^{\text{indicial eq}}$$

ie unless $n = \pm \ell$

ℓ solution

$$a_{\ell+2} = \frac{-a_\ell}{2 \cdot (2\ell+2)}, \qquad a_{\ell+4} = \frac{a_\ell}{2.4\ (2\ell+2)(2\ell+4)}$$

$$a_{\ell+2n} = \frac{(-)^n a_\ell}{2^{2n}\, n!\, (\ell+1)_n}$$

where $(\ell+1)_n \overset{\text{Pochhammer symbol}}{\equiv} \overbrace{(\ell+1)(\ell+2)\dots(\ell+n)}^{n \text{ factors}} = \dfrac{(\ell+n)!}{\ell!} = \dfrac{\Gamma(\ell+1+n)}{\Gamma(\ell+1)}$

$$a_\ell \sum \frac{(-)^n x^n}{2^{2n}\, n!\, (\ell+1)_n}$$

soln is $= \ell!\, a_\ell\, x^\ell \displaystyle\sum_{n=0}^{\infty} \frac{(-)^n (x/2)^{2n}}{n!\, (\ell+n)!} = \ell!\, a_\ell\, J_\ell(x)$

$-\ell$ solution

$$a_{-\ell+2} = \frac{-a_{-\ell}}{(-2\ell+2).\ 2}, \qquad a_{-\ell+4} = \frac{a_{-\ell}}{(-2\ell+2)(-2\ell+4).\ 2.4}$$

$$a_{-\ell+2n} = \frac{(-)^n a_{-\ell}}{2^{2n}\, n!\, \underbrace{(-\ell+1)_n}_{(-\ell+1)\dots(-\ell+n)}}$$

but blows up at $n = \ell$ if $\ell = n$

Is fine if ℓ is not an integer

$$\Gamma(-\ell+1)\, a_{-\ell}\, x^{-\ell} \sum_{n=0}^{\infty} \frac{(-)^n (x/2)^{2n}}{n!\, \Gamma(-\ell+1+n)} = \Gamma(-\ell+1)\, a_{-\ell}\, J_{-\ell}(x)$$

Q: How to find 2nd soln in general?

A: Wronskian.

Bessel equation

$$y'' + \frac{1}{x} y' + \left(1 - \frac{\ell^2}{x^2}\right) y = 0$$

has $p(x) = \frac{1}{x}$ $\quad q(x) = 1 - \frac{\ell^2}{x^2}$

Take $J_\ell(x)$ as first solution.

$$\begin{aligned} \ln W &= -\int p(x)\,dx + \text{const} \\ &= -\int \frac{dx}{x} + \text{const} \\ &= -\ln x + \text{const} \end{aligned}$$

$\Rightarrow W = \frac{\text{constant}}{x}$

take $W = \frac{1}{x}$ wlog.

$$\begin{aligned} y_2 &= y_1 \int \frac{W}{y_1^2}\,dx \\ &= J_\ell(x) \left[\int \frac{dx}{x \cdot J_\ell(x)^2} + \text{const} \right] \\ &= \frac{\pi}{2} Y_\ell(x) + \text{const}\ J_\ell(x) \end{aligned}$$

Eigenfunction methods for linear Diff Eqs

Show BHFS spher harmonics.

Perturbations (waves)
and wavefunctions in QM
satisfy linear Diff Eqs.
Powerful approach is to expand perts/wfs
in complete sets of eigenfunctions.
Why? Because eigenfunctions evolve
independently, without interfering.

Important examples:

1. Fourier modes. Simple harmonic.
   Eigenmodes of ~~translation~~ momentum operator $p = i\frac{d}{dx}$
   ~~Perts~~ Waves on uniform background - sound - water - light.
   Your ear resolves sound waves
   into Fourier modes.
2. Spherical harmonics. (Associated) Legendre.
   Eigenmodes of rotation operator $\vec{L} = \vec{r} \times \vec{p}$.
   Perts on spherically symmetric background.
   - CMB
   - helioseismology
   - magnetic or gravitational field around a planet
3. Bessel modes.
   Eigenmodes of radial part of Laplacian

$$\nabla^2 = \underbrace{\frac{1}{r^2}\frac{\partial}{\partial r} \frac{r^2 \partial}{\partial r}}_{\text{radial}} - \underbrace{\frac{L^2}{r^2}}_{\text{angular}}$$

Perts on uniform background when you want to expand modes into radial and angular parts.

4. Logarithmic radial modes.
   Eigenmodes of $i\frac{\partial}{\partial \ln r}$.

   Alternative radial / angular split of Laplacian
   $$\nabla^2 = \frac{1}{r^2}\left(\frac{\partial^2}{\partial \ln r^2} + \frac{\partial}{\partial \ln r} + L^2\right)$$
   I've found these useful in characterizing galaxy clustering at large scales, where fluctuations are linear.
5. Hydrogenic wavefunctions.
   (Associated) Laguerre.
6. Simple harmonic oscillator wavefunctions.
   Hermite.

All above are special cases of hypergeometric functions.

Eigenfunctions

An eigenfunction y of a linear differential operator $\mathcal{L}$ is a soln of

$$\mathcal{L} y = \underset{\substack{\uparrow \\ \text{eigenvalue.}}}{\lambda} y$$

analogous to matrix eqn

$$\sum_j \underset{\substack{\uparrow \\ \text{matrix}}}{\mathcal{L}_{ij}} \underset{\substack{\uparrow \\ \text{eigenvector}}}{y_j} = \underset{\substack{\uparrow \\ \text{eigenvalue}}}{\lambda_i} \underset{\substack{\uparrow \\ \text{eigenvector.}}}{y_i}$$

To motivate how to proceed, let's review some matrix theory...

Hermitian matrices

In matrix theory, Hermitian matrices are of fundamental importance because of 3 theorems:

1. Eigvals of Hermitian $\mathcal{L}$ are real
2. Eigvecs —— are orthogonal
3. Eigvecs —— form complete set.

Q: What is a Hermitian matrix $\mathcal{L}$?

A: One whose Hermitian conjugate is itself

$$\mathcal{L}^{\dagger} = \mathcal{L}$$

dagger signifies Herm conj

Q: What is the Herm conj of a matrix $\mathcal{L}$?

$$\mathcal{L}^{\dagger} = \underbrace{(\mathcal{L}^{T})}_{\text{transpose}}{}^{*} = \text{complex conjugate transpose}$$

complex conj

ie $\mathcal{L}^{\dagger}_{ij} = \mathcal{L}^{*}_{ji}$ in component form.

Q: Is Hermitian mx square?
A: Yes.
Q. If $\mathcal{L}$ is Hermitian and also real, what kind of matrix is it?
A: Symmetric.
Q: If $\mathcal{L}$ is symmetric and complex, is it Hermitian?
A: No.
Q: If $\mathcal{L}$ is not Hermitian, are theorems 1-3 true?
A: Generally, no.

Theorems 1-2 are fairly easy to prove, so we shall do so. Informs eigfn approach.
Theorem 3 is not at all easy.

Proof of thms 1 & 2.
Let y_a, y_b be eigvecs of Hermitian $\mathcal{L}$

$$\begin{pmatrix} \mathcal{L} \end{pmatrix}\begin{pmatrix} y_a \end{pmatrix} = \lambda \begin{pmatrix} y_a \end{pmatrix}$$

$$\mathcal{L} y_a = \lambda_a y_a \qquad \text{(no implicit summation)}$$
$$\mathcal{L} y_b = \lambda_b y_b$$

Consider

$$(y_a^*)\begin{pmatrix} \mathcal{L} \end{pmatrix} \quad y_a^\dagger \mathcal{L} \underset{\text{Herm}}{\overset{\mathcal{L}}{=}} y_a^\dagger \mathcal{L}^{\dagger\dagger} = (\overbrace{\mathcal{L} y_a}^{\lambda_a y_a})^\dagger = \overset{\lambda_a^\dagger}{\overset{\downarrow}{\lambda_a^*}} y_a^\dagger$$

So

$$(y_a^*)\begin{pmatrix} \mathcal{L} \end{pmatrix}\begin{pmatrix} y_b \end{pmatrix} \quad y_a^\dagger \mathcal{L} y_b = y_a^\dagger (\overbrace{\mathcal{L} y_b}^{=\lambda_b y_b}) = \lambda_b y_a^\dagger y_b$$
$$= (\underbrace{y_a^\dagger \mathcal{L}}_{\lambda_a^* y_a^\dagger}) y_b = \lambda_a^* y_a^\dagger y_b$$

ie

$$(\lambda_b - \lambda_a^*) y_a^\dagger y_b = 0.$$

Q: What kind of object is $y_a^\dagger \mathcal{L} y_b$? Matrix? Vector? Number?
A: Number, Complex.

Thm 1:

Take $a = b$:

$$(\lambda_a - \lambda_a^*) \underbrace{y_a^\dagger y_a} = 0$$

Q: What kind of object is $y_a^\dagger y_a$?

A: Number. Q: Real, complex, ...?

A: Positive real.

$$= \sum_i y_a^{i*} y_a^i = \sum_i |y_a^i|^2 > 0 \text{ for non-triv } y_a$$

$\Rightarrow \lambda_a = \lambda_a^* \Rightarrow \lambda_a$ is real QED

Thm 2:

Q: What kind of object is $y_a^\dagger y_b$?

A: Number.

Q: Real, complex, ...?

A: Complex.

eg $y_a = z\, y_b$ some complex number.

Take $a \neq b$

$$(\lambda_b - \lambda_a^*) y_a^\dagger y_b = 0$$

$\lambda_a^* = \lambda_a$ as λ_a is real

$\Rightarrow \lambda_b = \lambda_a$ or $\underbrace{y_a^\dagger y_b = 0}$

is statement that y_a and y_b are orthog.

ie eigvecs assoc with distinct eigvals are necessarily orthogonal.

If $\lambda_b = \lambda_a$ (= "degenerate" eigenvals)

then possible that $y_a^\dagger y_b = c \neq 0$.

Then define new eigvec

$$y_b - \frac{c\, y_a}{y_a^\dagger y_a} \leftarrow \text{a constant}$$

which is orthog to y_a:

$$y_a^\dagger \left(y_b - \frac{c\, y_a}{y_a^\dagger y_a} \right) = c - c\, \frac{y_a^\dagger y_a}{y_a^\dagger y_a} = 0 \quad ✓.$$

Q: could new eigvec be zero?

A: no, if y_a & y_b are lin indept

So eigvecs assoc with degen eigvals may not be orthogonal, but they can always be chosen to be so.

By normalizing eigvecs approproprately can always take eigvecs y_a to form an orthonormal set

$$y_a^+ y_b = \delta_{ab}$$

Thm 3 (Courant & Hilbert)
Eigvecs form complete set.
Q: What does "complete set" mean?
A: Any vector ψ may be expressed as a linear combination of a complete set y_a of vectors:

$$\psi = \sum_a u_a y_a$$

~~Prove by induction:~~
~~(a) Prove that $n \times n$ Hermitian matrix has at least 1 eigenvalue & eigenvector.~~
~~(b) Take $(n-1)$-dimensional space orthog to eigenvector, yields $(n-1) \times (n-1)$ Herm mx~~
~~(c) Done by induction.~~

Proof of Thm 3

Actually, thm 3 ain't always true! But it's true for an important class of operators, namely those that are bounded below (or above), meaning

$$\langle y | \mathcal{L} | y \rangle \geq \lambda_{min}$$

for all functions y, for some real λ_{min}.

Proof:

1. Choose y_0 satisfying $\langle y_0 | \mathcal{L} | y_0 \rangle = \lambda_{min}$.
2. Show that space of funcs u orthog to y_0, $\langle u | y_0 \rangle = 0$, satisfies $\langle u | \mathcal{L} | y_0 \rangle = 0$.
3. Hence conclude that $\mathcal{L} y_0 = \lambda_{min} y_0$ ie. y_0 is an eigenfunc.
4. Iterate to countably ∞ number of eigfuncs.
5. Consider $v \equiv u - \sum_{n=0}^{N} u_n y_n$. Show that $\langle v \| v \rangle \to 0$ as $N \to \infty$.

What about uncountably ∞ number of eigenfuncs?

Eg — real space eigfns $y_a = \delta(x - x_a)$

— Fourier — — $y_k = \frac{1}{\sqrt{2\pi}} e^{ikx}$

These are attainable as limit of operators with discrete spectrum of eigvals.

Thm Two Hermitian operators with complete set of eigfuncs have simultaneous eigenfuncs $\Leftrightarrow$ the two operators commute.

Proof $\Rightarrow$: For each eigfn y_n,

$$A B y_n = A b_n y_n = b_n A y_n = b_n a_n y_n$$
$$B A y_n = B a_n y_n = a_n B y_n = a_n b_n y_n$$

So $[A,B]\, y_n \equiv (AB - BA)\, y_n \equiv 0 \quad \forall\, n.$

Since eigfns form complete set,

$$[A,B]\, u = 0 \qquad \forall\, u$$

$$\Rightarrow [A,B] = 0$$

Conversely $\Leftarrow$:

Let y_n be eign of A, $\quad A y_n = a_n y_n$.

Then $B y_n = \sum_m b_{nm} y_m$ by completeness.

Then

$$0 = [A,B]\, y_n = (AB - BA)\, y_n$$
$$= A \sum_m b_{nm} y_m - B a_n y_n$$
$$= \sum_m b_{nm} (a_m - a_n)\, y_m$$

But y_n are orthog, so

$$b_{nm}(a_m - a_n) = 0 \qquad \forall\, m$$

For eigvals $a_m \neq a_n$,

then $b_{nm} = 0$.

For eigvals $a_m = a_n$ and $m \neq n$

Eigfn methods for linear Diff Eqs, Part 2.

Let $\mathcal{L}$ be a linear ordinary differential operator.

$$\mathcal{L} = p_n(x)\frac{d^n}{dx^n} + \ldots + p_1(x)\frac{d}{dx} + p_0(x).$$

Q: In what sense is this linear?

A: $\mathcal{L}(y_1(x) + y_2(x)) = \mathcal{L}y_1(x) + \mathcal{L}y_2(x)$.

Analogously to matrix theory, Hermitian diff ops are of fundamental importance because of 3 theorems Q: Which are?

1. Eigvals of Hermitian $\mathcal{L}$ are real
2. Eigvecs ———————— are orthogonal
3. Eigvecs ———————— form complete set.

Q: Examples?

A: Fourier modes, spherical harmonics, ...

Q: What is a Hermitian diff op $\mathcal{L}$?

A: Well, let's see.

What ingredients are needed?

1. Those needed to make the theorems work.

Ingredient 1 (the most fundamental ingredient):
Inner product of functions

With matrices, the eigenvectors y_a of a Hermitian operator were orthogonal if (Q?)

$$\underbrace{y_a^\dagger}_{\text{row matrix}} \underbrace{y_b}_{\text{column mx}} = 0 \quad \text{for} \quad a \neq b$$

$$\begin{pmatrix} y_a^* \end{pmatrix} \begin{pmatrix} y_b \end{pmatrix}$$

$\dagger = T*$

Herm conj = transpose complex conjugate

An eigenvector y_a is normalized if (Q?)

$$y_a^\dagger y_a = 1 .$$

Enshrine this by defining
Scalar product of vectors u & v

$$u \cdot v \equiv u^\dagger v = \sum_i u_i^* v_i$$

$(u^*)\begin{pmatrix} v \end{pmatrix}$ in component notation.

Analogously for functions, define inner product of functions $u(x)$ and $v(x)$

$$u \cdot v \equiv u^\dagger v \equiv \underbrace{\langle u|}_{\text{bra}} \underbrace{|v\rangle}_{\text{ket}} \equiv \int_{x_-}^{x_+} u^*(x)\, v(x)\, dx$$

Dirac notation

$\langle u| = u^\dagger$

$|v\rangle = v$

ie $\sum_i \rightsquigarrow \int dx$

Slight subtlety:
More generally

$$u \cdot v = \int_{x_-}^{x_+} u^*(x)\, v(x)\, \underbrace{w(x)}_{\text{weight function}} dx$$

ie $\sum_i \rightsquigarrow \int w(x)\, dx$

For example, spherical harmonics $Y_{\ell m}(\theta,\phi)$ are orthonormal over solid angle

$$\int_{\text{sphere}} Y^*_{\ell' m'}(\theta,\phi)\, Y_{\ell m}(\theta,\phi)\, \underbrace{d\Omega}_{\substack{= \sin\theta\, d\theta\, d\phi \\ = \text{interval of solid angle}}}$$

$$= \delta_{\ell'\ell}\, \delta_{m'm} = \begin{cases} 1 & \text{if } \ell'=\ell \text{ and } m'=m \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{\text{sphere}} ?\, d\Omega = \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} ?\, \underbrace{\sin\theta}_{\substack{\text{weight} \\ \text{function}}}\, d\theta\, d\phi$$

However, can always eliminate weight function by a change of variable

$$\int_{\theta=0}^{\pi} \ldots \sin\theta\, d\theta = \int_{x=-1}^{1} \ldots\, dx \qquad x \equiv \cos\theta$$

Q: Where did − sign go in $d\cos\theta = -\sin\theta\, d\theta$?

A: Swapped limits of integration from $\begin{matrix}\theta = 0 \text{ to } \pi \\ x = 1 \text{ to } -1\end{matrix}$ to $x = -1$ to 1.

So can take weight function $= 1$ wlog.

We will take weight fn to be 1 or not 1 according to our convenience.

Hilbert space

Is defined to be an infinite dimensional vector space equipped with an inner product.

Can regard $u(x)$ as a vector $\underset{\substack{\uparrow \\ \text{index of vector}}}{u_x}$ in a Hilbert space.

Slick trick, as we'll see.

Ingredient 2 in definition of Hermitian linear operator:

Adjoint linear operator $\mathcal{L}^+$

Proof of thms 1 & 2 for matrices

$\underset{\substack{\text{eigvals}\\ \text{real}}}{\uparrow}$ $\underset{\substack{\text{eigvecs}\\ \text{orthog}}}{\uparrow}$ for Hermitian $\mathcal{L}$

involved looking at

$$y_a^+ \mathcal{L}\, y_b$$

$$(\ y_a^*\ )\Big(\ \mathcal{L}\ \Big)\Big(y_b\Big)$$

Q: What kind of object is this?

A: Number, complex.

Functional equivalent is

$$y_a^+ \mathcal{L}\, y_b \equiv \langle y_a | \mathcal{L} | y_b \rangle$$

$$\equiv \int_{x_-}^{x_+} y_a^*(x)\, \mathcal{L}\, y_b(x)\, dx$$

With matrices considered

$$y_a^+ \mathcal{L}\, y_b = y_a^+(\mathcal{L} y_b)$$
$$= (y_a^+ \mathcal{L}^+)\, y_b \quad \text{for } \mathcal{L} = \mathcal{L}^+$$

Q: What kind of matrix is $\mathcal{L} = \mathcal{L}^+$?

A: Hermitian.

Q: Is the adjoint the same as the Hermitian conjugate?

A: Yes. But by convention a Hermitian operator is also assoc with boundary conditions. Somewhat analogous to Green's func.

[GF is the inverse $G = \mathcal{L}^{-1}$ of a diff op, but requires b.cs.]

What does $(y_a^{+} \mathcal{L}^{+})\, y_b$ mean applied to functions?
It's

$$(y_a^{+} \mathcal{L}^{+})\, y_b = \int_{x_-}^{x_+} \underbrace{(y_a^{+} \mathcal{L}^{+})}_{= (\mathcal{L} y_a)^{+}}\, y_b \, dx$$

so $\mathcal{L}^{+}$ is a differential operator that somehow acts on the thing to its left.
Can be accomplished by integrating by parts.
Take 2nd order linear ord diff op

$$\mathcal{L} = p_2(x) \frac{d^2}{dx^2} + p_1(x) \frac{d}{dx} + p_0(x)$$

Integrate by parts:

$$y_a^{+} \mathcal{L}\, y_b$$

$$= \int_{x_-}^{x_+} y_a^{*}(x) \left[p_2 \frac{d^2}{dx^2} + p_1 \frac{d}{dx} + p_0 \right] y_b(x) \, dx$$

$$= \underbrace{\Big[\text{something} \Big]_{x_-}^{x_+}}_{\text{"surface" terms}} + \int_{x_-}^{x_+} \frac{d^2 p_2 y_a^{*}}{dx^2}\, y_b \, dx - \int_{x_-}^{x_+} \frac{d\, p_1 y_a^{*}}{dx}\, y_b \, dx + \int_{x_-}^{x_+} p_0 y_a^{*}\, y_b \, dx$$

$$= \text{surf terms} + \int_{x_-}^{x_+} \underbrace{\left[\left(\frac{d^2}{dx^2} p_2 - \frac{d}{dx} p_1 + p_0 \right) y_a^{*} \right]}_{\text{interpret this as desired } \mathcal{L}^{+}} y_b \, dx$$

So define adjoint $\mathcal{L}^+$ by

$$\mathcal{L}^+ \equiv \frac{d^2}{dx^2} p_2 - \frac{d}{dx} p_1 + p_0$$

Compared to $\mathcal{L}$,

(a) coeffs $p_i(x)$ have changed sides

$$p_i \frac{d^i}{dx^i} \to \frac{d^i}{dx^i} p_i$$

(b) odd derivs acquire $-$ sign from integration by parts.

Operator $\mathcal{L}$ is <u>self-adjoint</u> iff

$$\mathcal{L}^+ = \mathcal{L}$$

(not quite Hermitian, which also requires $\mathcal{L}$ to act on functions satisfying appropriate b.c.s).

Expand out $\mathcal{L}^+$

$$\mathcal{L}^+ = \frac{d}{dx}\left(p_2 \frac{d}{dx} + p_2'\right) - p_1 \frac{d}{dx} - p_1' + p_0$$

$$= p_2 \frac{d^2}{dx^2} + (2p_2' - p_1)\frac{d}{dx} + p_2'' - p_1' + p_0$$

this $= \mathcal{L}$ (ie $\mathcal{L}$ is self-adjoint) if

(a) $2p_2' - p_1 = p_1$ ie $p_2' = p_1$

(b) $p_2'' - p_1' + p_0 = p_0$ ie $p_2'' = p_1'$,
which follows from $p_2' = p_1$.

So $\mathcal{L}^+ = \mathcal{L}$ iff $p_2' = p_1$.

If so, then

$$\mathcal{L}^{\dagger} = \frac{d}{dx} p_2 \frac{d}{dx} + \cancelto{0}{(p_2' - p_1)} \frac{d}{dx} + \cancelto{0}{(p_2'' - p_1')} + p_0$$

$$= \frac{d}{dx} p_2 \frac{d}{dx} + p_0 \qquad \leftarrow \text{SL form}$$

$$= p_2 \frac{d^2}{dx^2} + p_2' \frac{d}{dx} + p_0 \qquad \leftarrow \text{SL form}$$

Last 2 versions are in "Sturm-Liouville" form. An S-L operator is just a self-adjoint operator.

Redo integration by parts for self-adjoint operator $\mathcal{L}$, but now keeping surface terms:

$y_a^{\dagger} \mathcal{L} y_b$

$$\equiv \int_{x_-}^{x_+} y_a^* \left(\frac{d}{dx} p_2 \frac{d}{dx} + p_0 \right) y_b \, dx$$

$$= \int_{x_-}^{x_+} y_a^* \frac{d}{dx}\left(p_2 \frac{dy_b}{dx} \right) dx + \int y_a^* p_0 y_b \, dx$$

$$= \left[y_a^* p_2 \frac{dy_b}{dx} \right]_{x_-}^{x_+} - \int_{x_-}^{x_+} \frac{dy_a^*}{dx} p_2 \frac{dy_b}{dx} dx + \int y_a^* p_0 y_b \, dx$$

$$= \left[y_a^* p_2 \frac{dy_b}{dx} \right]_{x_-}^{x_+} - \left[\frac{dy_a^*}{dx} p_2 y_b \right]_{x_-}^{x_+} + \int_{x_-}^{x_+} \frac{d}{dx}\left(p_2 \frac{dy_a^*}{dx} \right) y_b \, dx$$

$$+ \int_{x_-}^{x_+} y_a^* p_0 y_b \, dx$$

$$= \left[y_a^* \, p_2 \, y_b' \right]_{x_-}^{x_+} - \left[y_a^{*\prime} \, p_2 \, y_b \right]_{x_-}^{x_+} + \underline{\int_{x_-}^{x_+} (\mathcal{L} y_a^*) \, y_b \, dx}$$

$$= (\mathcal{L} y_a)^\dagger \, y_b$$

$$= y_a^\dagger \mathcal{L}^\dagger y_b$$

This $= y_a^\dagger \mathcal{L}^\dagger y_b$
provided that surface terms vanish.

Hence definition of
<u>Hermitian diff op $\mathcal{L}$</u>
$\mathcal{L}$ is a Hermitian operator
with respect to a set of functions $y_a(x)$
if
(1) $\mathcal{L}$ is self-adjoint, $\mathcal{L}^\dagger = \mathcal{L}$
(2) y_a satisfy boundary conditions
at x_-, x_+, such that surface
terms vanish.

Why this definition?
Because the 3 theorems apply
(with qualifications ...) to Hermitian op $\mathcal{L}$.
We'll get back to the 3 theorems,
but first, the most important
example of a Hermitian diff op
with a complete set of eigfuncs
is ... (Q?)

Fourier modes

eigenmodes of

$$\frac{d^2y}{dx^2} = -k^2 y$$

Here $\mathcal{L} = \frac{d^2}{dx^2}$, eigvals $\lambda = -k^2$.

Eigenmodes are

$$y_k \propto e^{\pm ikx}$$

Fourier modes come in 3 flavors:

1. Fourier series:
   k discrete, x continuous,
   y(x) periodic
2. Continuous Fourier modes
   k and x both continuous, real
3. Discrete Fourier modes
   k and x both discrete.
   Fast Fourier Transform (FFT) method yields exact discrete Fourier transform.

In literature, some authors treat Fourier transform as synonymous with FFT. It is NOT.