ASTR 5110 Atomic and Molecular Processes Fall 2022. Problem Set 5. Due Wed 5 Oct

1. Deuterium

(a) Energy levels

Deuterium D is a stable isotope of hydrogen H, having a nucleus containing one proton and one neutron. Use the Bohr model to calculate the shift

$$\frac{E_{\rm D} - E_{\rm H}}{E_{\rm H}} \tag{1.1}$$

of energy levels of D relative to those of H. [Hint: The energy is proportional to the reduced mass of the two-body system. You may approximate $m_{\rm D} = 2m_{\rm H}$.]

(b) Velocity shift

Hence infer the velocity shift, in km/s, of lines of D relative to H. Are the D lines redshifted or blueshifted (longer or shorter wavelength) relative to H? Are lines of D observationally distinguishable from those of H?

(c) Does it matter?

Amazingly, the observed abundance of D to H inferred from absorption by intergalactic gas against quasars at high redshift agrees well with the predictions of Big Bang Nucleosynthesis in the Standard Model of Cosmology. The observed primordial D/H ratio by mass (not number!) is (Tanabashiet al. (Particle Data Group), Phys. Rev. D98, 030001, 2018)

$$\frac{\rm D}{\rm H} = (2.57 \pm 0.03) \times 10^{-5} \ . \tag{1.2}$$

Have a look on the arXiv (search for deuterium) to find *one* other circumstance in astrophysics or planetary science where deuterium plays an important role despite its small abundance relative to hydrogen. Explain, in a few sentences. Please reference the arXiv paper you consulted.

2. (De)-excitation of levels of Hydrogen

(a) Collisional cross-sections

Argue from the Bohr model that the cross-section σ for (de)-excitation of energy level n of hydrogen by collisions with electrons approximates

$$\sigma \approx \pi n^4 a_0^2 \ . \tag{2.1}$$

(b) Collisional cross-sections

Because of the large collisional cross-sections at the high n indicated by equation (2.1), collisional excitation and ionization, and their inverses collisional de-excitation and three-body recombination, dominate the population of the highest-n levels of hydrogen,

$$e + {\rm H}_L \leftrightarrow e + {\rm H}_U \ , \quad e + {\rm H} \leftrightarrow 2e + p \ , \tag{2.2}$$

where L and U denote respectively lower and upper levels of H. Argue that these processes are liable to drive the high-n levels of H into relative thermodynamic equilibrium at the electron temperature. What does that imply about the relative number densities n_U/n_L of the energy levels of H? [Hint: Apologies that n is, conventionally, used to indicate both the energy level n and the number density n. Don't be confused.]

(c) Collisional excitation vs. de-excitation

Detailed balance implies that in thermodynamic equilibrium the rates of collisional excitation and de-excitation of levels of H by electrons are equal,

$$n_e n_L C_\uparrow = n_e n_U C_\downarrow \ , \tag{2.3}$$

where n_e, n_L, n_U are number densities, and $C_\uparrow$ and $C_\downarrow$ are rate coefficients. Argue that the ratio of collisional excitation to de-excitation rates by a Maxwellian distribution of electrons is therefore

$$\frac{C_\uparrow}{C_\downarrow} = \frac{g_U}{g_L} e^{-\Delta E/kT} \ , \tag{2.4}$$

where $\Delta E \equiv E_U - E_L$ is the difference in energy levels. Does the relation (2.4) hold if the electrons do not have a Maxwellian distribution? Does some version of the relation (2.4) hold between individual angular-momentum substates of n-levels?

(d) Collisional ionization vs. three-body recombination

Detailed balance implies that in thermodynamic equilibrium the rates of collisional ionization and three-body recombination are equal,

$$n_e n_L C_\uparrow = n_e^2 n_p C_\downarrow \ , \tag{2.5}$$

where $C_\uparrow$ and $C_\downarrow$ are rate coefficients for collisional ionization and three-body recombination. Use the Saha equation to deduce the ratio $C_\uparrow/C_\downarrow$ of the rate coefficients of ionization to three-body recombination by a Maxwellian distribution of electrons. Does the formula for $C_\uparrow/C_\downarrow$ hold if the electrons do not have a Maxwellian distribution?

(e) Radiative versus collisional de-excitation

The dipole transition formula implies that the rate coefficient A for allowed spontaneous radiative decay out of energy level n of hydrogen (to any other level) approximates, in atomic units,

$$A \approx \alpha^3 n^{-5} \ , \tag{2.6}$$

where $\alpha \approx 1/137$ is the fine-structure constant. Estimate the energy level n at which collisional and radiation transition rates are approximately equal,

$$n_e \sigma v \approx A \ . \tag{2.7}$$

Ignore radiative excitation (appropriate in low density environments). You should find an expression that depends on $n_e v$. At approximately what $n_e v$ are essentially all energy levels in relative thermodynamic equilibrium? Qualitatively, what happens to the populations of low-n levels if $n_e v$ is less than this? [Hint: First use equations (2.1) and (2.6) to estimate the energy level n with $n_e v$ expressed in atomic units. Then convert from atomic units to SI units. If you like, you can take a typical electron temperature to be $10^4\,\mathrm{K}$, equivalent to $\approx 1\,\mathrm{eV}$, or approximately $1/20$ of an atomic unit of energy.]