Radiative Processes

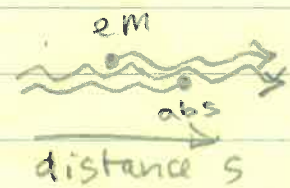
Electromagnetic radiation at $t, \vec{x}$ characterized by intensity

$$I_\nu = \frac{\text{energy}}{\text{area . time . freq . solid angle}} \qquad \text{eg } \frac{\text{erg}}{\text{cm}^2 \text{ s . Hz . sr}}$$

sr ← steradian

Governed by
Radiative transfer equation

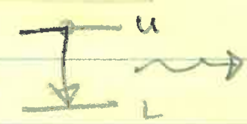
$$\frac{\partial I_\nu}{\partial s} = \underbrace{j_\nu}_{\text{source}} - \underbrace{k_\nu I_\nu}_{\text{sink}}$$



j_ν is emissivity ; k_ν is opacity.

(i) Emissivity from spontaneous emission

$$\frac{\text{phots emitted}}{\text{vol . time}} = n_u A_{uL}$$



n_u: number density in upper state cm^{-3}

A_{uL}: "Einstein coefficient" for spontaneous emission s^{-1}

$$j_\nu = \frac{\overbrace{h\nu}^{\text{photon energy}}}{\underbrace{4\pi}_{\text{all directions}}} \phi_\nu \, n_u A_{uL} \qquad \frac{\text{energy}}{\text{vol . time . freq . ster}}$$

ϕ_ν: line profile normalized so $\int \phi_\nu \, d\nu = 1$

Note $\phi_\nu \, d\nu = \phi_\nu \, \nu \, d\ln\nu = \phi_\nu \, \nu \frac{dv}{c} = \frac{\phi_\nu}{\lambda} dv$ (velocity)

(ii) <u>Opacity from absorption</u>

$$\frac{\text{photons absorbed}}{\text{vol. time}} = n_L \, I_{\nu_{UL}} \, B_{LU}$$

same units as $n_U A_{UL}$ B_{LU}: Einstein coefficient for absorption.

$$\kappa_\nu^{abs} = n_L \sigma_\nu = \frac{h\nu}{4\pi} \phi_\nu \, n_L B_{LU}$$

σ_ν: cross-section at freq ν

= inverse mean free path of photon

units $\frac{1}{\text{length}}$

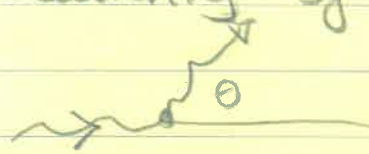
Note: Opacity in stars is conventionally defined as $\rho \sigma_\nu$

ρ: mass density instead of numb density.

<u>Scattering</u> = absorption followed by emission at same freq.

Ex/ Dipole scattering of unpolarized light

see (10.15)



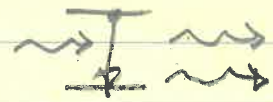
$$\kappa_\nu^{dip} \propto \frac{3}{16\pi}\left(1 + \cos^2\theta\right) \qquad \frac{3}{16\pi}\int (1+\cos^2\theta)\, d\Omega = 1$$

e.g. (unpolarized) Thomson
Rayleigh
~ any non-rel

} scattering

(iii) Negative opacity from stimulated emission

$$\frac{\text{phots emitted}}{\text{vol . time}} = n_u I_{\nu_{uL}} B_{uL}$$



$$\kappa_\nu = \frac{h\nu}{4\pi}\phi_\nu \left(\underbrace{n_L B_{Lu}}_{\text{abs}} - \underbrace{n_u B_{uL}}_{\text{stim emission}}\right)$$

Einstein relations

In TE. $I_\nu \overset{?}{=} B_\nu(T)$ = $\dfrac{2h\nu^3}{c^2}\overbrace{\dfrac{1}{e^{h\nu/kT}-1}}^{\text{occup. no.}}$

Planck function

Detailed balance :

$2h\nu . c \dfrac{p^2 dp d\Omega}{h^3} = 2h\nu c \left(\dfrac{h}{c}\right)^3 \dfrac{1}{h^3}$ $\nu^2 d\nu$

$p = \frac{h\nu}{c}$

$$\frac{n_L}{n_u} = \frac{g_L}{g_u} e^{-h\nu_{uL}/kT}$$

$$n_u A_{uL} = B_{\nu_{uL}}\left(n_L B_{Lu} - n_u B_{uL}\right)$$

$$\cancel{n_u} A_{uL} = \frac{2h\nu_{uL}^3}{c^3\left(e^{h\nu/kT}-1\right)} \cancel{n_u} \frac{\left(\frac{g_L}{g_u} e^{h\nu_{uL}/kT} B_{Lu} - B_{uL}\right)}{e^{h\nu_{uL}/kT} - 1}$$

$$\Rightarrow \boxed{g_L B_{Lu} = g_u B_{uL} = \frac{c^2}{2h\nu^3} g_u A_{uL}} \quad \text{Einstein relations}$$

Another way of writing detailed balance is

$$j_\nu = B_\nu \kappa_\nu$$

which $\Rightarrow$ same thing.

Opacity

Einstein relations

$$\Rightarrow \quad \kappa_\nu = \frac{h\nu}{4\pi}\phi_\nu\, n_L B_{Lu}\Big(\underset{\text{abs}}{1} - \underset{\text{stim. em}}{b_{uL}\, e^{-h\nu/kT}}\Big)$$

$$b_{uL} \equiv \frac{n_u/n_L}{n_u/n_L|_{TE}} = \text{departure coefficient} = 1 \text{ in TE.}$$

Examples

(1) "Nebular approximation"
$\sim$ all atoms in ground state $\quad n_u \ll n_L$
$\Rightarrow$ ignore stim. em

(2) $h\nu \gg kT$
eg. optical/UV absorption lines
in cool absorbing cloud $\Rightarrow$ ignore stim. em

(3) 21 cm HI line

F=1 triplet
F=0 singlet

Collisional exc & de-exc of hyperfine levels of ground state much faster than rad. decay
$\Rightarrow$ levels in TE at kinetic temperature T.
But $\overset{21\text{cm}}{h\nu} \ll \overset{\text{kinetic}}{kT}$
$\Rightarrow \quad 1 - e^{-h\nu/kT} \approx h\nu/kT$

$$\Rightarrow \quad \kappa_\nu = \frac{h\nu}{4\pi}\phi_\nu\, n_L B_{Lu}\frac{h\nu}{kT}$$

(4) Population inversion.
If $\dfrac{n_u}{n_L} > \dfrac{g_u}{g_L}$ $\quad \left(\text{NOT } \dfrac{n_u}{n_L} > \dfrac{n_u}{n_L}\Big|_{TE}\right)$
then $\kappa_\nu < 0$
$\Rightarrow$ masering (will do later).
Easier to obtain when $h\nu \ll kT$ (radio).

Oscillator strength

Absorption oscillator strength f_{LU}

dimensionless

"number of harmonic oscillators"

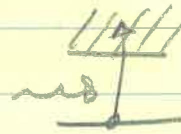
~~remember dipole em formula~~?

defined by

$$\frac{h\nu}{4\pi} B_{LU} = \frac{\pi e^2}{m_e c} f_{LU} = \int \sigma_\nu \, d\nu$$

For continuum absorption

$$\frac{\pi e^2}{m_e c} df = \sigma_\nu \, d\nu$$



photoionization cross-section

Emission oscillator strength f_{UL} defined by

$$-\frac{h\nu}{4\pi} B_{UL} = \frac{\pi e^2}{m_e c} f_{UL}$$

ie. (Einstein) $\quad f_{UL} = -\frac{g_L}{g_U} f_{LU}$ negative.

Conventionally "osc. str." refers to abs. osc. str.

Osc. str. sum rule

$$\sum_{u} f_{LU} = \text{number of electrons in atom}$$

u = all bound (discrete) and free (cts) states including those below L, for which f_{LU} is −ve

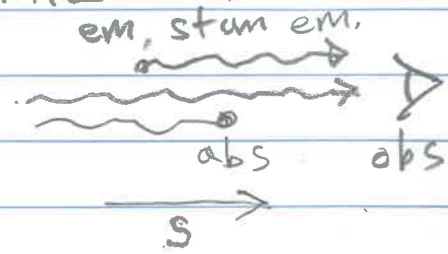
$$= \begin{cases} 1 & \text{H-like} \\ 2 & \text{He-like} \\ 3 & \text{Li-like} \\ \text{etc} & \end{cases}$$

Proof: Landau & Lifshitz QM eq 149.10

R & L eq (10.36)

Radiative transfer equation RTE

$$\frac{\partial I_\nu}{\partial s} = \underset{\text{spont. em}}{j_\nu} - \underset{\text{abs \& stim. em}}{\kappa_\nu I_\nu}$$



or $\dfrac{\partial I_\nu}{\kappa_\nu \partial s} = \dfrac{j_\nu}{\kappa_\nu} - I_\nu$

Define "optical depth"

$\tau_\nu \equiv \int \kappa_\nu \, ds$ = number of mean free paths of photon between source & obs

and "source function"

$$S_\nu \equiv \frac{j_\nu}{\kappa_\nu}.$$

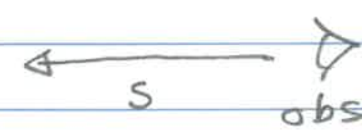
Then $\dfrac{\partial I_\nu}{\partial \tau_\nu} = S_\nu - I_\nu$.

In T.E., $S_\nu = I_\nu = B_\nu(T)$.
Planck function

(Formal) solution of RTE

Common to ray trace backwards from observer to source, so

$\underset{\text{forwards along ray}}{s} \rightarrow \underset{\text{backwards along ray}}{-s}$, $\tau_\nu \rightarrow -\tau_\nu$.



Then $-\dfrac{\partial I_\nu}{\partial \tau_\nu} = S_\nu - I_\nu$

$$-\frac{\partial I_\nu}{\partial \tau_\nu} + I_\nu = S_\nu$$

$$= -e^{\tau_\nu} \frac{\partial}{\partial \tau_\nu}\left(e^{-\tau_\nu} I_\nu\right)$$

$$\Rightarrow -\frac{\partial}{\partial \tau_\nu}\left(e^{-\tau_\nu} I_\nu\right) = e^{-\tau_\nu} S_\nu$$

$$\Rightarrow -\left[e^{-\tau_\nu} I_\nu\right]_{obs}^{\infty} = \int_{obs}^{\infty} e^{-\tau_\nu} S_\nu \, d\tau_\nu$$

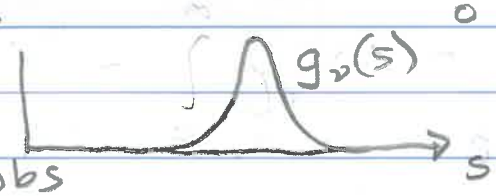
$$\tau_\nu \rightarrow \begin{cases} 0 & obs \\ \infty & \infty \end{cases}$$

$$= I_\nu(obs).$$

Introduce "visibility function"

$$g_\nu(s) \equiv e^{-\tau_\nu} \frac{d\tau_\nu}{ds}$$

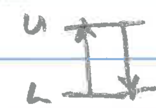
satisfies $\int_{obs}^{\infty} g_\nu(s)\, ds = \int_0^{\infty} e^{-\tau_\nu} d\tau_\nu = 1$



Then $\boxed{I_\nu(obs) = \int_{obs}^{\infty} g_\nu(s)\, S_\nu(s)\, ds = \underbrace{\langle S_\nu(\text{source}) \rangle}_{\text{average around } \tau_\nu \sim 1}}$

Example: S_ν dominated by single transition

$$S_\nu \equiv \frac{j_\nu}{\kappa_\nu} = \frac{\frac{h\nu}{4\pi} \phi_\nu \underbrace{n_u A_{uL}}_{em}}{\frac{h\nu}{4\pi} \phi_\nu \left(\underbrace{n_L B_{Lu}}_{abs} - \underbrace{n_u B_{uL}}_{stim\ em} \right)}$$



assume L, u same velocity distn $\Rightarrow \phi_\nu$ same

$$= \frac{2h\nu^3}{c^2} \frac{1}{\left(\underbrace{\frac{n_L/g_L}{n_u/g_u}}_{= e^{h\nu/kT_{ex}}} - 1 \right)}$$

defines exc. temp of $L \Leftrightarrow u$ transition

$$= B_\nu(T_{ex})$$

$$\Rightarrow I_\nu(\text{obs}) = \underset{\text{av. around } \tau_\nu \sim 1}{\langle B_\nu(T_{ex}) \rangle}$$

More generally, multiple transitions and continuum may contribute at any ν.

Local thermodynamic equilibrium LTE

Simplest approximation for stellar atmosphere.

Assume:
- velocity distributions
- energy levels
- ionization

of all atoms (and molecules) in TE at temperature $T_{ex}(s)$ at height s.

Radiation is NOT in TE.

Then $I_\nu(\text{obs}) = \underset{\text{av. around } \tau_\nu \sim 1}{\langle B_\nu(T_{ex}) \rangle}$.

Q: Doesn't intensity I_ν decrease as r^{-2} at distance r from source?

Q: Look at photosphere with hi-res telescope. Do you see B_ν or dilute B_ν?

Brightness temperature

If $h\nu \ll kT$ (typically true in radio)

$$B_\nu(T) = \frac{2\nu^2 kT}{c^2} \quad \text{"Rayleigh-Jeans" regime}$$

defines brightness temperature T.

RTE solves to

$$T_\nu = \underset{\text{av. around } \tau_\nu \sim 1}{\langle T_{ex} \rangle}$$

General time-dependent RTE

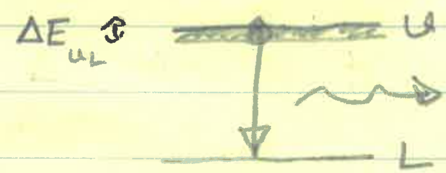
Boltzmann equation eg. Cosmic Mic Background.

Line profiles ϕ_ν

Sources of line-broadening:

1. Natural broadening } Lorentz profile
2. Collisional/pressure broadening } Lorentz profile
3. Thermal broadening } Gaussian profile
4. Turbulent broadening } Gaussian profile
5. Voigt profile = convolution of Lorentz & Gaussian profiles.

1. Natural broadening



Uncertainty principle

$$\underset{\substack{\uparrow \\ \Delta h\nu_{UL}}}{\Delta E_{UL}} \quad \underset{\substack{\uparrow \\ ? \\ \text{lifetime}}}{\Delta t} \approx \hbar \qquad \text{lifetime} \sim \frac{1}{A_{UL}}$$

More generally, lifetime Δt depends on sum over all transition:

$$\Gamma_{UL} = \underbrace{\sum_{i<U} A_{Ui}}_{\text{width of upper state}} + \underbrace{\sum_{i<L} A_{Li}}_{\text{width of lower state}}$$

Spose know atom is in state U at $t=0$.
Expect state to decay with

probability in U $= e^{-\Gamma t}$.

Emitted radiation field has wave-function

$$\psi \propto e^{-2\pi i \nu_{UL} t} \, e^{-\frac{\Gamma}{2} t} \qquad (t > 0)$$

(square to get probability).

To get frequency ν distribution,
Fourier transform:

$$\psi_\nu \propto \int_0^\infty \underbrace{e^{-2\pi i \nu_{UL} t - \frac{\Gamma}{2} t}}_{\psi(t)} e^{2\pi i \nu t} \, dt$$

$$= \left[\frac{e^{(2\pi i(\nu - \nu_{UL}) - \frac{\Gamma}{2})t}}{2\pi i(\nu - \nu_{UL}) - \frac{\Gamma}{2}} \right]_0^\infty$$

$$= \frac{1}{2\pi i(\nu - \nu_{UL}) - \frac{\Gamma}{2}}$$

Line profile ϕ_ν is

$$\phi_\nu \propto |\psi_\nu|^2 \propto \frac{1}{(2\pi(\nu - \nu_{UL}))^2 + \left(\frac{\Gamma}{2}\right)^2}$$

Normalize to $\int_{-\infty}^{\infty} \phi_\nu \, d\nu = 1$

Note $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \left[\tan^{-1} x\right]_{-\infty}^{\infty} = \pi$

$$\boxed{\phi_\nu = \frac{1}{\pi} \frac{\Gamma/4\pi}{(\nu - \nu_{UL})^2 + (\Gamma/4\pi)^2}}$$ Lorentz profile

2. Collisional / pressure broadening

Collisions (de)excite atomic state, shortening its lifetime.
Still expect exponential decay out of state.
Result is same Lorentz profile, but

$$\Gamma_{UL} = \underbrace{\Gamma_U}_{\text{inv. lifetime of } U} + \underbrace{\Gamma_L}_{\text{inv. lifetime of } L}$$

from all processes.

3. Thermal broadening

In ap. Non-relativistic atoms & ions almost invariably have momentum distrns in TE (why?) ie Maxwell-Boltzmann distrns.

Line-of-sight velocity distrn (why line of sight)

$$f(v_z)\,dv_z \propto e^{-mv_z^2/2kT}\,dv_z$$

mass of what? ion

Doppler shift $\dfrac{\Delta\nu}{\nu_{UL}} = \dfrac{v_z}{c}$

$\Rightarrow$ Gaussian line profile

$$\phi_\nu = \frac{1}{\sqrt{\pi}\,\Delta\nu_{th}} \exp\left[-\frac{(\nu-\nu_{UL})^2}{\Delta\nu_{th}^2}\right]$$

Note $\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}$

where $\dfrac{\Delta\nu_{th}}{\nu_{UL}} = \dfrac{\Delta v_{th}}{c} = \dfrac{\left(\dfrac{2kT}{m}\right)^{1/2}}{c} \propto \left(\dfrac{T}{m}\right)^{1/2}$

Heavier ions have narrower lines

4. Turbulent broadening

There are environments, such as molecular clouds in ISM, which show random velocities in excess of thermal broadening. Often modelled as Gaussian with

Δv_{turb} in place of Δv_{th}

$\Delta \nu_{turb}$ $\qquad\qquad$ $\Delta \nu_{th}$

Heavier ions? Same line widths.

Why Gaussian?

Central Limit Theorem states, sum of large number of "random variables" drawn from same (not nec Gaussian) prob distrn is Gaussian

Write Δv_D for empirical "Doppler" width

$\Delta \nu_D$ agnostic of origin.

arXiv:1310.8569 Appendix Fig 1, 2
astro-ph/0302006 Kirkman et al D/H.

Voigt profiles of resonance absorption lines

in ISM/IGM, Eg. "Ly α forest" against QSOs.
For resonance abs by "cool" intervening gas, can ignore emission, stim. emission.

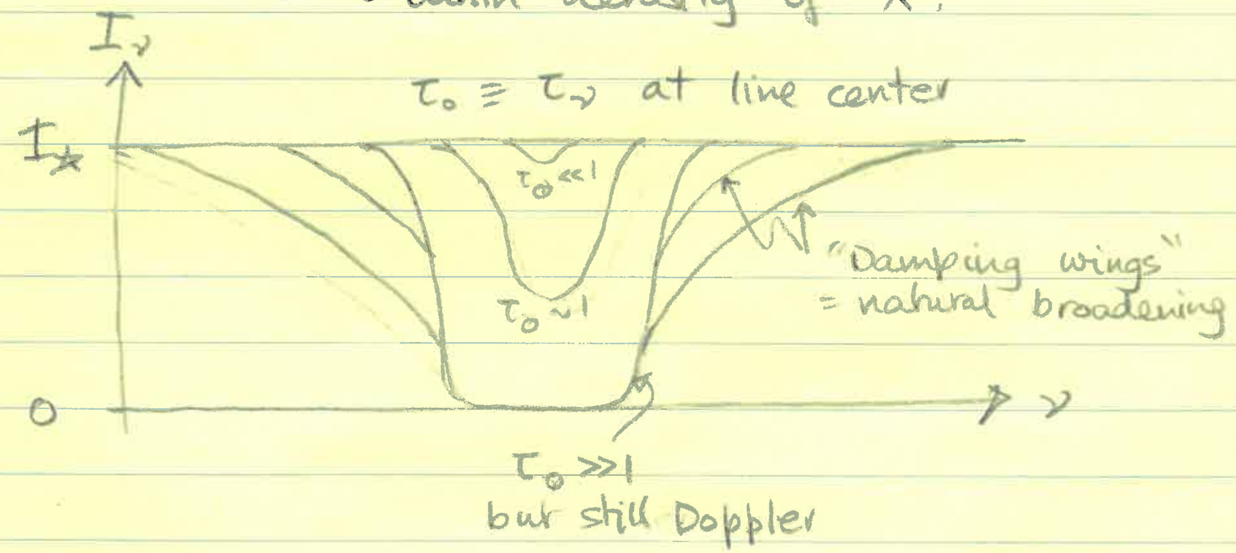
$$I_\nu = \overbrace{I_\star}^{\text{source intensity}} e^{-\tau_\nu} = I_\star e^{-t\phi_\nu}$$

where $t = \int\limits_{\text{obs}} \tau_\nu \, d\nu = N_x \left(\frac{\pi e^2}{m_e c}\right) \underbrace{f_{Lu}}_{\text{(abs). osc. str.}}$

$$N_x = \int n_x \, ds$$

$\star$ ↑ # density of absorbing species X

= column density of X.



Equivalent width

Often characterize strength of abs. line by its "equivalent width"

$$W_\nu \equiv \int\limits_{\text{line}} \frac{I_\star - I_\nu}{I_\star} \, d\nu \qquad \text{units? freq.}$$

$$= \int_{-\infty}^{\infty} \left(1 - e^{-\tau_\nu}\right) d\nu$$

If you prefer to plot spectrum
vs wavelength λ
or velocity v
instead of frequency ν

$$\frac{d\lambda}{\lambda_{u\ell}} = \frac{dv}{c} = \frac{d\nu}{\nu_{u\ell}}$$ (when valid? lines narrow)

$$\Rightarrow \frac{W_\lambda}{\lambda_{u\ell}} = \frac{W_v}{c} = \frac{W_\nu}{\nu_{u\ell}} = \text{(dimensionless eq width)}$$

(Dimensionless) eq. width in various regimes

$\tau_0 \ll 1$

$$\frac{W_\nu}{\nu_{u\ell}} \approx \int \tau_\nu \, d\nu / \nu_{u\ell} \qquad \frac{1}{\nu_{u\ell}} = \frac{\lambda_{u\ell}}{c}$$

$$= N_x \left(\frac{\pi e^2}{m_e c^2}\right) f_{\ell u} \lambda_{u\ell}$$

$$\propto N_x f_{\ell u} \quad \text{(abs) osc. str}$$

$$\frac{c}{\Delta v_D} \qquad = \frac{t}{\nu_{u\ell}} \frac{1}{\sqrt{\pi}} \frac{\nu_{u\ell}}{\Delta \nu_D} = \frac{t \lambda_{u\ell}}{\sqrt{\pi} \Delta v_D}$$

$\tau_0 \gg 1$ but still Doppler

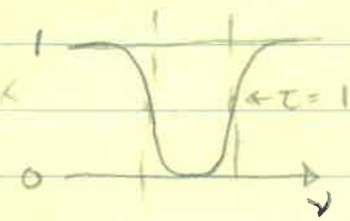
$$\frac{W_\nu}{\nu_{u\ell}} = \int_{-\infty}^{\infty} \left[1 - \exp\left(- \underbrace{t \frac{1}{\sqrt{\pi} \Delta \nu_D}}_{= \tau_0} e^{-\left(\frac{\Delta \nu}{\Delta \nu_D}\right)^2} \right) \right] \frac{d\nu}{\nu_{u\ell}}$$

$x \equiv \Delta \nu / \Delta \nu_D$

$$= \frac{\Delta \nu_D}{\nu_{u\ell}} \int_{-\infty}^{\infty} \left[1 - \exp\left(- \tau_0 e^{-x^2} \right) \right] dx$$

not doable by mathematica

$$\approx \frac{\Delta v_D}{c} \cdot 2 \int_0^{x_1} 1 \, dx$$



where $\tau_0 e^{-x_1^2} = 1$

ie $x_1 = \sqrt{\ln \tau_0}$

ie $\frac{W_\nu}{\nu_{u\ell}} \approx 2 \frac{\Delta v_D}{c} \sqrt{\ln \tau_0}$ for $\tau_0 \gg 1$

$\propto \sqrt{\ln N_x f_{\ell u}}$ $\qquad \tau_0 = N_x \left(\frac{\pi e^2}{m_e c^2}\right) f_{\ell u} \lambda_{u\ell} \frac{c}{\sqrt{\pi} \Delta v_D}$

$\tau_0 \gg 1$, damping wing

$$\frac{W_\nu}{\nu_{ul}} = \int_{-\infty}^{\infty} \left[1 - \exp\left(- t \, \frac{1}{\pi} \, \frac{(\Gamma/4\pi)}{\Delta\nu^2 + (\Gamma/4\pi)^2} \right) \right] \frac{d\nu}{\nu_{ul}}$$

$$\tau_0 = \frac{t}{\pi(\Gamma/4\pi)}$$

$$\approx \frac{2}{\nu_{ul}} \int_0^{\nu_1} 1 \, d\nu$$

where $\tau_0 \dfrac{(\Gamma/4\pi)^2}{\Delta\nu^2 + (\Gamma/4\pi)^2} = 1$

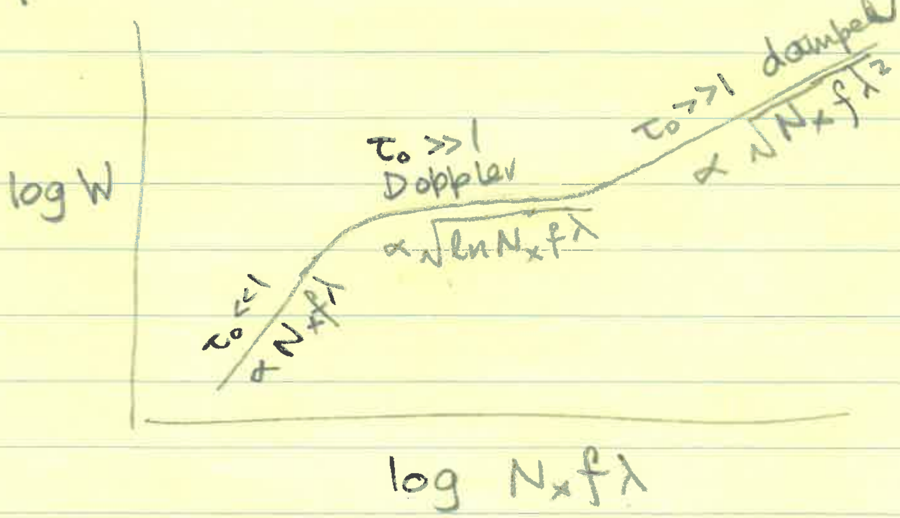
ie $\Delta\nu = \dfrac{\Gamma}{4\pi} \sqrt{\underbrace{\tau_0}_{\gg 1} - 1} \approx \dfrac{\Gamma}{4\pi}\sqrt{\tau_0} = \sqrt{\dfrac{t}{\pi}\dfrac{\Gamma}{4\pi}}$

$$= \sqrt{N_l \left(\frac{\pi e^2}{m_e c}\right) f_{lu} \frac{\Gamma}{4\pi^2}}$$

$$\frac{W_\nu}{\nu_{ul}} = \frac{2\Delta\nu}{\nu_{ul}} \propto \sqrt{N_x f_{lu} \lambda^2}$$

Curve of growth

Equiv width vs. column density



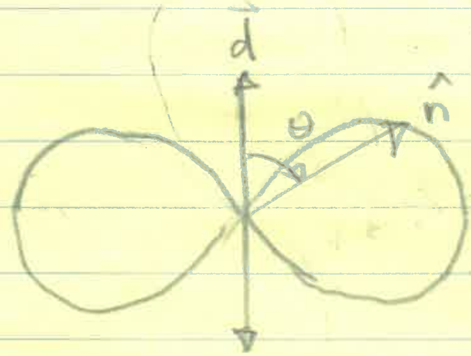
eg arXiv: 1209.0891 Fig 3.

Or DI abs lines in hi-z quasar.

Dipole emission pattern

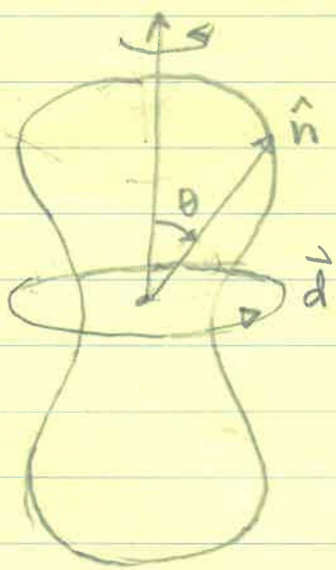
direction of emission — interval of solid angle.

$$\frac{dE}{dt} = \frac{(\hat{n} \times \ddot{\vec{d}})^2}{c^3} \frac{d\Omega}{4\pi}$$



$\vec{d}$ oscillating back & forth ↕
Emission pattern
azimuthally symmetric
about $\vec{d}$ direction

$$\frac{dE}{dt} \propto \sin^2\theta$$



$\vec{d}$ oscillating in circle
Emission pattern
az symmetric
about axis ⊥ to $\vec{d}$ plane

$$\frac{dE}{dt} \propto 1 + \cos^2\theta$$