Structure of non-rel QM

Wave-particle duality

Two slit experiment shows ptle-wave duality.

Google image?

	Ptle property		Wave property	
energy	E	$=$	$\hbar\omega$	$= h\nu$
momentum	$\vec{p}$	$=$	$\hbar\vec{k}$	$= \frac{hc}{\lambda}$ in magnitude

Exact or approximate?

QM asserts that wavefunction

$$\psi \propto e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

describes ptle(s) of definite energy-mom $(E, \vec{p})$.

Observables

In QM observables are operators that act on wavefunctions

Energy op $\hat{E} = i\hbar\frac{\partial}{\partial t}$

$$\hat{\vec{p}} = -i\hbar\frac{\partial}{\partial\vec{x}}$$

An observable has definite value if wavefunction is eigenstate

$$\underbrace{i\hbar\frac{\partial}{\partial t}}_{\text{operator}}\psi = \underbrace{E}_{\text{eigval}}\psi$$

$$-i\hbar\frac{\partial}{\partial\vec{x}}\psi = \vec{p}\,\psi$$

Many particle wavefunctions in non-rel QM

The wavefunc of a pair of particles is

$$\psi(t, \vec{x}_1, \vec{x}_2)$$

coords of ptle 1   coords of ptle 2.

If 2 particles are identical then under exchange of ptles

$$\psi(t, \vec{x}_2, \vec{x}_1) = \pm \psi(t, \vec{x}_1, \vec{x}_2)$$

bosons ($+$), fermions ($-$)

Note. In QFT (relativistic QM) this description fails, because it does not allow ptles to be created or annihilated.
In QFT, wavefuncs themselves become operators, that create & annihilate ptles out of the vacuum

$$\underbrace{\hat{\psi}^\dagger(t,\vec{x})}_{\text{creation op}} \underbrace{|0\rangle}_{\text{vac}} = \underbrace{\psi(t,\vec{x})}_{\text{wavefunc, or state}}$$

Hilbert space

In either non-rel QM or QFT, wavefunctions or quantum states belong to a Hilbert space, which
(1) is a finite or infinite dimensional complex vector space,
(2) equipped with a complex inner product

(1). Vector space means?

$$\underbrace{a}_{\text{complex number}} \underbrace{\psi}_{\text{vector}}, \quad \underbrace{\psi_1 + \psi_2}_{\text{vectors}} \quad \text{are vectors}$$

Inf-dim vec space huh?
Think of $\psi(\vec{x})$ as $\psi_{\vec{x}}$ (analogous to ψ_i)
vector with index $\vec{x}$.

Take Fourier transform of $\psi(\vec{x})$

$$\psi_{\vec{k}} = \psi(\vec{k}) = \int \psi(\vec{x}) e^{-i\vec{k}\cdot\vec{x}} d^3x$$

same vector in Hilbert space
expressed in "Fourier" instead
of "real" coordinates.

(2) Inner product means for any 2 wavefunctions
or "state vectors" ψ and ϕ,

$$\underset{\text{bra}}{\langle \psi} | \underset{\text{ket}}{\phi \rangle} = a \quad \text{is a complex number.}$$

Dirac notation

Analogous to scalar product of vectors in
Euclidean space (or in general relativity)
For single-particle wavefuncs

$$\langle \psi | \phi \rangle = \psi^{\vec{x}} \phi_{\vec{x}} = \int_V \psi^{+}(\vec{x}) \phi(\vec{x})\, d^3x \qquad \text{fixed time } t.$$

$$= \psi^{\vec{k}} \phi_{\vec{k}} = \int_{V_k} \psi^{+}(k) \phi(\vec{k}) \frac{d^3k}{(2\pi)^3} \qquad \text{"Parseval's theorem"}$$

$$\psi^{+}(\vec{x}) = \left(\cdots \psi_{\vec{x}} \cdots \right)^{*} \qquad \phi(\vec{x}) = \begin{pmatrix} \vdots \\ \phi_{\vec{x}} \\ \vdots \end{pmatrix}$$

Hermitian conjugate
= complex conjugate transpose

Note: In QFT $\psi^{+} \rightsquigarrow$ creation op
$\phi \rightsquigarrow$ annihilation op.

Observables and probabilities

Any observable (energy, mom., ang, mom) is described by a linear, Hermition operator A that has a complete set of orthonormal eigenfunctions ψ_n with real eigvalues a_n

$$\underset{\text{nth eigenstate}}{A\,\psi_n} = \underset{\text{nth eigenvalue}}{a_n}\,\psi_n$$

Orthonormal means

$$\langle \psi_m | \psi_n \rangle = \delta_{mn} \equiv \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

Complete means any wavefunction ψ can be expanded as linear combination

$$\psi = \sum_n \underset{\text{complex numbers}}{c_n}\,\psi_n \quad \text{of eigenstates } \psi_n$$

Orthonormality $\Rightarrow$

$$\langle \psi_n | \psi \rangle = c_n$$

c_n is amplitude to find ψ in state ψ_n

$|c_n|^2$ is probability " " " " "

Q: What does "prob to find ψ in state ψ_n" mean?

$$\langle \psi | \psi \rangle = \left(\sum_m c_m^* \psi_m^\dagger\right)\left(\sum_n c_n \psi_n\right)$$

$$= \sum_n |c_n|^2$$

$$= 1$$

says ψ is in state ψ with probability 1.

The mean value of property associated with operator A is

$$\langle A \rangle = \sum_n a_n |c_n|^2 = \langle \psi | A | \psi \rangle .$$

Hermitian operators

The Hermitian conjugate A^+ of op A is defined such that

$$(A\phi)^+ = \phi^+ A^+ \qquad \forall \text{ states } \phi$$

or

$$\langle A\phi| = \langle\phi|A^+$$

Operator A is Hermitian if

$$A = A^+$$

or equivalently

$$\langle\phi|A|\phi\rangle = \langle\phi|A^+|\phi\rangle \qquad \forall\ \phi$$

Properties of Hermitian operators:

(1) eigenvalues a_n are real

$$A\psi_n = a_n\psi_n$$
$$= A^+\psi_n = a_n^*\psi_n$$

(2) eigenstates with different eigenvalues are necessarily orthogonal

$$\langle\psi_m|A^+|\psi_n\rangle = \langle A\psi_m|\psi_n\rangle$$
$$= a_m\langle\psi_m|\psi_n\rangle$$
$$= \langle\psi_m|A|\psi_n\rangle = \langle\psi_m|A\psi_n\rangle$$
$$= a_n\langle\psi_m|\psi_n\rangle$$
$$\Rightarrow \langle\psi_m|\psi_n\rangle = 0 \quad \text{if} \quad a_m \neq a_n$$

Commutation of operators

$$[A, B] \equiv AB - BA$$

If A, B are 2 lin Herm ops, then they are simultaneously observable, ie have simultaneous eigenfuncs, if and only if

$$[A, B] = 0$$

Proof $\Rightarrow$;

Spose A B have simultaneous eigfns ψ_n.

$$AB\,\psi_n = A\,b_n\psi_n = a_n b_n \psi_n$$
$$= b_n a_n \psi_n = b_n A \psi_n$$
$$= BA\,\psi_n$$

$\Rightarrow$ $AB = BA$

$\Leftarrow$; TIY (see notes eq (5.4))

try