ASTR 3740 Relativity & Cosmology Spring 2025. Problem Set 4.
Due Wed 5 Mar

Warning: this problem set is quite lengthy, so please do not wait until the last day to start it.

1. Trajectories of particles in the Schwarzschild geometry

In this problem you will find it helpful to visit John Walker's web site at

http://www.fourmilab.to/gravitation/orbits/ .

The most fun part of the site is the Java applet, so you will probably want to seek out a Java-enabled computer, although you can also use John Walker's site without Java.

In what follows, the time t, radial coordinate r, polar angle θ, and azimuthal angle ϕ are the usual Schwarzschild coordinates in the Schwarzschild metric (with $c = 1$ as usual)

$$ds^2 = -\left(1 - \frac{r_s}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{r_s}{r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \ , \tag{1.1}$$

with r_s the Schwarzschild radius

$$r_s = 2GM \ . \tag{1.2}$$

Without loss of generality, the trajectory of a particle falling freely in the Schwarzschild geometry may be taken to lie in the equatorial plane, $\theta = \pi/2$. For a particle of finite (nonzero) mass, the trajectory satisfies the equations

$$\left(1 - \frac{r_s}{r}\right) \frac{dt}{d\tau} = E \ , \tag{1.3a}$$

$$r^2 \frac{d\phi}{d\tau} = L \ , \tag{1.3b}$$

$$\left(\frac{dr}{d\tau}\right)^2 + V_{\text{eff}} = E^2 \ , \tag{1.3c}$$

where τ is the proper time of the particle, and E and L are constants, the particle's energy and angular momentum per unit mass. The quantity V_{eff} is the effective potential given by

$$V_{\text{eff}} = \left(1 - \frac{r_s}{r}\right)\left(1 + \frac{L^2}{r^2}\right) \ . \tag{1.4}$$

(a) (3 points) Check

Are John Walker's equations the same as the ones given above (aside from possible differences in notation)?

(b) (3 points) Velocity at infinity

Argue from equations (1.3) that relative to the rest frame of the Schwarzschild geometry, the radial velocity $v_r \equiv dr/dt$ and the transverse velocity $v_\perp \equiv r d\phi/dt$ (the $\equiv$ sign means "is defined to be equal to") of the particle at extremely large distances from the Schwarzschild geometry, $r \to \infty$, are related to E and L by

$$v_r^2 = 1 - \frac{1}{E^2} - \frac{L^2}{E^2 r^2} , \tag{1.5a}$$

$$v_\perp = \frac{L}{Er} , \tag{1.5b}$$

(note that L can be extremely large at large r, so L/r is not necessarily zero in the limit $r \to \infty$). Hence show that the velocity $v_\infty \equiv (v_r^2 + v_\perp^2)^{1/2}$ of the particle as $r \to \infty$ is related to its energy E by

$$E = \frac{1}{(1 - v_\infty^2)^{1/2}} . \tag{1.6}$$

What does it mean if $E < 1$?

(c) (3 points) Extrema of the effective potential

Find the radii at which the effective potential V_{eff} is a maximum or a minimum, i.e. $dV_{\text{eff}}/dr = 0$, as a function of angular momentum L. You should find that extrema exist only if the absolute value $|L|$ of the angular momentum exceeds a certain critical value L_c. What is that critical value?

(d) (3 points) Sketch

Sketch what the effective potential looks like for values of L (i) less than, (ii) equal to, (iii) greater than the critical value L_c. Make sure to label the axes clearly and correctly. Describe physically, in words, what the possible orbital trajectories are for the various cases. [Hint: You will need to experiment with different choices of axes to make the graph look good. I found it clearer *not* to start the effective potential at zero. For cases (i) and (iii), values near the critical value L_c showed the distinction most clearly.]

(e) (3 points) Circular orbits

Circular orbits, satisfying $dr/d\tau = 0$, occur where the effective potential is a minimum (stable orbit) or a maximum (unstable orbit). Show (from your equation for the extrema of the effective potential) that the angular momentum L of a particle in circular orbit at radius r satisfies

$$|L| = \frac{r}{\left(\dfrac{2r}{r_s} - 3\right)^{1/2}} , \tag{1.7}$$

and hence show also that the energy E in this circular orbit is

$$E = \frac{2^{1/2}(r - r_s)}{[r(2r - 3r_s)]^{1/2}} . \tag{1.8}$$

(f) (3 points) Orbital period

Show that the orbital period t, as measured by an observer at rest at infinity, of a particle in circular orbit at radius r is given by Kepler's 3rd law (yes, it's true even in the fully general relativistic case!)

$$\frac{GMt^2}{(2\pi)^2} = r^3 \ . \tag{1.9}$$

[Hint: The time measured by an observer at rest at infinity is just the Schwarzschild time t. Argue that the azimuthal angle ϕ evolves according to

$$\frac{d\phi}{dt} = \frac{L(r - r_s)}{Er^3} \ . \tag{1.10}$$

The period t is the time taken for ϕ to change by 2π.]

(g) (3 points) Infall time

Calculate the proper time τ for a particle with $L = 0$ and $E = 1$ to fall from a finite radius r to the singularity at zero radius. What is the physical significance of the choice $L = 0$ and $E = 1$? [Hint: Write down the equation for $dr/d\tau$ for $L = 0$ and $E = 1$, and then solve it.]

(h) (3 points) Infall time — numbers

Use your answer to part (g) to show that the proper time to fall from the Schwarzschild radius $r = r_s$ to the singularity (for $L = 0$ and $E = 1$) is, in units including c,

$$\tau = \frac{4GM}{3c^3} \ . \tag{1.11}$$

Evaluate your answer, in seconds, for the case of a black hole of mass $4 \times 10^6 \ \mathrm{M}_\odot$, such as may be in the center of our Galaxy, the Milky Way. [Constants: https://physics.nist.gov/cuu/Constants/ $c = 299{,}792{,}458 \ \mathrm{m\,s^{-1}}$; $G = 6.6743 \times 10^{-11} \ \mathrm{m^3\,kg^{-1}\,s^{-2}}$; $1 \ \mathrm{M}_\odot = 1.99 \times 10^{30} \ \mathrm{kg}$.]

2. Photons in the Schwarzschild geometry

The orbit equations (1.3) would appear to break down for photons, which have zero mass, hence infinite energy per unit mass E (cf. equation [1.6] for $v_\infty = 1$) and infinite angular momentum per unit mass L. Another way of looking at this is that photons follow null geodesics, $d\tau = 0$, so that τ, which does not change, is not a very useful time coordinate for expressing the equations of motion of photons.

The difficulty is cured by introducing an "affine parameter" $\lambda = E\tau$, which functions as a good scalar coordinate along null geodesics. In terms of the affine parameter λ, the equations of motion (1.3) for freely falling massless particles, such as photons, become

$$\left(1 - \frac{r_s}{r}\right) \frac{dt}{d\lambda} = 1 \ , \tag{2.1a}$$

$$r^2 \frac{d\phi}{d\lambda} = J \ , \tag{2.1b}$$

$$\left(\frac{dr}{d\lambda}\right)^2 + \mathcal{V}_{\text{eff}} = 1 \ , \tag{2.1c}$$

where $J = L/E$ is the photon's angular momentum per unit energy, and $\mathcal{V}_{\text{eff}} = V_{\text{eff}}/E$ is the effective potential given by

$$\mathcal{V}_{\text{eff}} = \left(1 - \frac{r_s}{r}\right) \frac{J^2}{r^2} \ . \tag{2.2}$$

(a) (6 points) Circular orbits

Circular orbits, occur where the effective potential $\mathcal{V}_{\text{eff}}$ is a minimum (stable orbit) or a maximum (unstable orbit). At what radius can photons orbit in circles? Is the orbit stable or unstable?