ASTR 3740 Relativity & Cosmology Spring 2023. Problem Set 7.
Due Wed 26 Apr

1. Horizon Problem

(a) Expansion factor

The temperature of the CMB today is $T_0 \approx 3\,\mathrm{K}$. By what factor has the Universe expanded (i.e. what is a_0/a) since the temperature was the GUT (Grand Unified Theory) temperature $T \approx 10^{29}\,\mathrm{K}$? [Hint: Argue that $T \propto a^{-1}$ during the expansion of the Universe.]

(b) Hubble distance

By what factor has the Hubble distance c/H increased during the expansion of part (a)? Assume that the Universe has been mainly radiation-dominated during this period, and that the Universe is flat. [Hint: For a flat Universe $H^2 \propto \rho$, and for radiation-dominated Universe $\rho \propto a^{-4}$.]

(c) Comoving Hubble distance

Hence determine by what factor the comoving Hubble distance $x_H = c/(aH)$ has increased during the expansion of part (a).

(d) Comoving Hubble distance during inflation

During inflation the Hubble distance c/H remained constant, while the cosmic scale factor a expanded exponentially. What is the relation between the comoving Hubble distance $x_H = c/(aH)$ and cosmic scale factor a during inflation? [You should obtain an answer of the form $x_H \propto a^{?}$.]

(e) e-foldings to solve the Horizon Problem

By how many e-foldings must the Universe have inflated in order to solve the Horizon Problem? Assume again, as in part (a), that the Universe has been mainly radiation-dominated during expansion from the GUT temperature to the current temperature, and that this radiation-dominated epoch was immediately preceded by a period of inflation. [Hint: Inflation solves the Horizon Problem if the currently observable Universe was within the Hubble distance at the beginning of inflation, i.e. if the comoving $x_{H,0}$ now is less than the comoving Hubble distance $x_{H,\mathrm{i}}$ at the beginning of inflation. The 'number of e-foldings' is $\ln(a_\mathrm{f}/a_\mathrm{i})$, where ln is the natural logarithm, and a_i and a_f are the cosmic scale factors at the beginning (i for initial) and end (f for final) of inflation.]

2. Flatness Problem

An amusing statement of this cosmological problem can be found on Ned Wright's graph at https://astro.ucla.edu/~wright/cosmo_03.htm#FO .

(a) Friedmann's equation

Use the definitions $H^2 = \frac{8}{3}\pi G\rho_c$ of the critical density ρ_c, and $\Omega \equiv \rho/\rho_c$ of Omega, to show that Friedmann's equation (including the curvature term)

$$H^2 = \frac{8}{3}\pi G\rho - \frac{\kappa c^2}{a^2} \tag{2.1}$$

can be rewritten in the form

$$\frac{\Omega - 1}{\Omega} = \frac{3\kappa c^2}{8\pi G\rho a^2} \ . \tag{2.2}$$

(b) Evolution of Ω with a

Suppose once again that $\rho \propto a^{-n}$. Show that (a simple consequence of eq. (2.2))

$$\frac{\Omega - 1}{\Omega} \propto a^{?} \tag{2.3}$$

where ? is an exponent which you should derive (in terms of n).

(c) Here's the flatness problem

Suppose that the temperature at the moment of the Big Bang was about the GUT temperature $\sim 10^{29}$ K. The radiation temperature of the Universe today is of course that of the CMB, about 3 K. If Ω_0 (subscript 0 means the present time) is of order, but not equal to, one at the present time ($\Omega_0 \sim 0.3$, say), roughly how close to one was Ω at the Big Bang? [Hint: Define the small quantity $\epsilon \equiv \Omega - 1$, and use (2.3) to estimate ϵ at the Big Bang. Note that for tiny ϵ, you can approximate $1 + \epsilon \approx 1$. Assume that $T \propto a^{-1}$ during the expansion of the Universe, and for simplicity that the Universe has been radiation-dominated for most of that expansion, so that $n \approx 4$.]

(d) Relation between Horizon and Flatness Problems

Show that Friedmann's equation (2.1) can be written in the form

$$\Omega - 1 = \kappa x_H^2 \tag{2.4}$$

where $x_H \equiv c/(aH)$ is the comoving Hubble distance. Use this equation to argue in your own words how the horizon and flatness problems are related. [The main part of this question is not the math but the explanation. You should convince the grader that you understand physically what is going on.]