ASTR 3740 Relativity & Cosmology Spring 2023. Problem Set 6.
Due Wed 19 Apr

1. Anti-gravity

(a) Condition for an accelerating Universe

Suppose that the Universe contains only matter energy (M) and vacuum energy (a cosmological constant Λ), and that it is geometrically flat

$$\Omega_M + \Omega_\Lambda = 1 \tag{1.1}$$

where $\Omega_M \equiv \rho_M/\rho_c$ and $\Omega_\Lambda \equiv \rho_\Lambda/\rho_c$ are the contributions to Omega in matter and vacuum. How big must Ω_Λ be for the Universe to be accelerating? [Hint: Friedmann's equation for the acceleration $\ddot{a} \equiv d^2a/dt^2$ of the cosmic scale factor $a(t)$ is

$$\frac{\ddot{a}}{a} = -\frac{4}{3}\pi G(\rho + 3p) \tag{1.2}$$

which shows that the Universe is accelerating if $\rho + 3p < 0$. Ordinary matter has mass-energy density ρ_M but essentially no pressure, $p_M = 0$, while vacuum has negative pressure equal to its mass-energy density, $p_\Lambda = -\rho_\Lambda$.]

(b) Draw your own conclusion

The final (2018) analysis of data from the Planck satellite, coupled with other CMB data, supernovae, galaxy clustering, and other astrophysical data, indicates $\Omega_M = 0.3$ and $\Omega_\Lambda = 0.7$ (https://arxiv.org/abs/1807.06209). Is our Universe accelerating?

2. Solutions to Friedmann's equations in a Flat Universe

Suppose that the Universe is flat, $\kappa = 0$, so that Friedmann's energy equation reduces to

$$\frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho\ . \tag{2.1}$$

Suppose further that the Universe is dominated by stuff whose mass-energy density ρ varies with cosmic scale factor a as

$$\rho \propto a^{-n} \tag{2.2}$$

as the Universe expands, with n a constant. For example, $n = 3$ for ordinary matter, $n = 4$ for radiation, and $n = 0$ for vacuum energy.

(a) Case $n \neq 0$

Solve Friedmann's equation to show that, for $n \neq 0$,

$$a \propto t^{2/n}\ . \tag{2.3}$$

[Hint: You should find that Friedmann's equation can be recast in the form $t = \int f(a)da$ where $f(a)$ is some function of cosmic scale factor a. You may set $a = 0$ at $t = 0$, which says that the Universe had zero size at zero age.]

(b) Deceleration or acceleration?

For what range of n is the Universe decelerating ($\ddot{a} < 0$) or accelerating ($\ddot{a} > 0$)? Is the Universe decelerating or accelerating in the particular cases of a matter-dominated ($n = 3$) or radiation-dominated ($n = 4$) Universe?

(c) Case $n = 0$

The case $n = 0$ corresponds to vacuum density, which remains constant as the Universe expands. Solve Friedmann's equation for this case to show that

$$a \propto e^{Ht} \tag{2.4}$$

where $H \equiv \dot{a}/a$, the Hubble constant, is in this case a constant in time as well as space. What is the Hubble constant H here in terms of the vacuum energy ρ_Λ?

(d) For your information (no credit)

You may be wondering whether there is a relation between the index n in this question and the pressure p in the Anti-Gravity question. The answer is yes. It is straightforward to show (but I'm not asking you to do this) from the energy equation $d(\rho a^3) + p d(a^3) = 0$ (which you may recognize as the equation $dE + p dV = 0$ of thermodynamics) that

$$n = 3\left(1 + \frac{p}{\rho}\right) \ . \tag{2.5}$$

3. Phyics Nobel Prize in Astrophysics & Cosmology

Has the Physics Nobel Prize ever been awarded for work in astrophysics or cosmology? If so, to whom, and for what? [Look it up on the web — and don't forget to refererence your sources.]