ASTR 3740. Spring 2007.
Using the River Model to Draw Geodesics around Black Holes

Fancy creating a computer program, maybe a Java applet, to draw the orbits of particles or photons around black holes? The river model provides a nice way to implement this.

Before doing so, you should be aware that general relativity admits arbitrary coordinate systems, and the coordinate system of the river model is only one such coordinate system. However, some coordinate systems reveal the physics more clearly than others, and I think that the river model offers the most intuitively appealing coordinate system for black holes.

Because the coordinate systems of general relativity are arbitrary, the orbits you draw cannot be considered as an absolute representation of reality. In general relativity one says that the representation of orbits is "gauge-dependent", i.e. it depends on the choice of coordinate system. Nevertheless, the orbits you obtain in the river model will show all the correct orbital structure: stable and unstable orbits, the photon sphere, and so on.

An example of a gauge-independent representation of orbits would be to show what an observer would actually see watching a particle free fall around a black hole. However, this would require not only following the orbit of the particle, but also ray-tracing light from the particle to the observer, an altogether more complex problem.

1. Qualitative Summary

The river model of black holes is a mathematically rigorous description of stationary black holes of arbitrary mass, electric charge, and spin. In the river model, space itself flows like river through a flat background, while objects move through the river according to the rules of special relativity.

For spherical black holes, the river falls radially inward, hitting the speed of light at the horizon. Inside the horizon, the river falls faster than light, carrying everything with it.

For rotating black holes, one might have anticipated that the river would spiral inwards, but this is not the case: the azimuthal component of the river velocity is zero. Instead, the river possesses at each point not only a velocity, but also a twist. That is, the river has a Lorentz structure, characterized by six numbers (velocity and rotation), not just three (velocity). As an object moves through the river, it is Lorentz boosted by tidal changes in the velocity of the river along its path, and spatially rotated by tidal changes in the twist of the river along its path.

To implement this on a computer, set up a system of flat background coordinates, and attach an imaginary inflowing river of space to the flat background. Then shoot particles, each with whatever initial 4-velocity you like relative to the inflowing river. At each frame, update the position of a particle according to its 4-velocity, and update the 4-velocity of a particle by applying a small Lorentz transformation. The desired Lorentz transformation is given by the tidal change in the velocity and twist fields of the river along the path of the particle. That's it!

2. Spherically Symmetric Black Hole

Use geometric units $c = G = 1$. For a spherically symmetric black hole of mass M and electric charge Q the river metric is, in spherical coordinates $(t_{\rm ff}, r, \theta, \phi)$,

$$\mathrm{d}s^2 = -\,\mathrm{d}t_{\rm ff}^2 + (\mathrm{d}r + \mathrm{v}\,\mathrm{d}t_{\rm ff})^2 + r^2(\mathrm{d}\theta^2 + \sin^2\!\theta\,\mathrm{d}\phi^2) \tag{1}$$

where v is the radial infall river velocity

$$\mathrm{v} = \frac{(2Mr - Q^2)^{1/2}}{r} \tag{2}$$

and $t_{\rm ff}$ is the proper time experienced by an object that free falls radially inward from zero velocity at infinity. The river velocity is positive for a black hole (infalling), negative for a white hole (outfalling). Horizons occur where the river velocity v equals the speed of light

$$\mathrm{v} = \pm 1 \tag{3}$$

with $\mathrm{v} = 1$ for black hole horizons, and $\mathrm{v} = -1$ for white hole horizons. The river metric (1) was first pointed out in 1921, independently by Gullstrand and by Painlevé, for the case of the Schwarzschild geometry (uncharged spherical black hole).

Rewrite the river metric (1) in Cartesian coordinates $x^\mu \equiv (x^0, x^1, x^2, x^3) \equiv (t_{\rm ff}, x, y, z)$ with origin at the center of the black hole:

$$\mathrm{d}s^2 = \eta_{\mu\nu}(\mathrm{d}x^\mu - \mathrm{v}^\mu \mathrm{d}t_{\rm ff})(\mathrm{d}x^\nu - \mathrm{v}^\nu \mathrm{d}t_{\rm ff}) \tag{4}$$

where $\eta_{\mu\nu} \equiv \mathrm{diag}\,(-1, 1, 1, 1)$ is the Minkowski metric, and v^μ are the components of the river velocity

$$\mathrm{v}^\mu = \mathrm{v}\left(0\,,\,-\frac{x}{r}\,,\,-\frac{y}{r}\,,\,-\frac{z}{r}\right)\ . \tag{5}$$

Introduce a test particle, either a massive particle or a photon, that free falls around the black hole. Later on you might consider allowing the particle to accelerate, perhaps by firing its rockets (or perhaps by being electrically charged, if the black hole is also charged), but for the moment let the test particle fall freely. Define τ to be the proper time experienced by the particle. In one frame of your computer animation, you will want to advance the proper time by some small constant interval $\delta\tau$. Let u^μ denote the 4-velocity of the test particle

$$u^\mu \equiv \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \equiv \left(\frac{\mathrm{d}t_{\rm ff}}{\mathrm{d}\tau}\,,\,\frac{\mathrm{d}x}{\mathrm{d}\tau}\,,\,\frac{\mathrm{d}y}{\mathrm{d}\tau}\,,\,\frac{\mathrm{d}z}{\mathrm{d}\tau}\right)\ . \tag{6}$$

If in one frame the proper time of the particle advances by $\delta\tau$, then the coordinates of the particle advance by

$$\delta x^\mu = u^\mu \delta\tau\ . \tag{7}$$

Observers who free-fall radially from zero velocity at infinity have 4-velocity

$$u^\mu_{\rm ff} = (1, \mathrm{v}^1, \mathrm{v}^2, \mathrm{v}^3)\ . \tag{8}$$

Such observers are comoving with the inflowing river of space. Introduce a set of locally inertial frames, tetrads, attached to such radially free-falling observers. Associated with the locally inertial frame at each point is a set of locally inertial coordinates $\xi^m \equiv (\xi^0, \xi^1, \xi^2, \xi^3)$. The convention adopted here is that latin indices m signify locally inertial tetrad frames, while greek indices μ signify curved coordinate frames. Latin indices m are raised and lowered with the Minkowski metric η_{mn}, while greek indices μ are raised and lowered with the coordinate metric $g_{\mu\nu}$. Whereas the coordinates x^μ are globally defined over the whole space, the locally inertial tetrad coordinates ξ^m are only locally defined within a small (infinitesimal) neighborhood of each point.

The 4-velocity υ^m of the test particle with respect to the locally inertial tetrad frame at the position of the particle is

$$\upsilon^m \equiv \frac{\mathrm{d}\xi^m}{\mathrm{d}\tau} \tag{9}$$

and is related to the coordinate 4-velocity u^μ by

$$\begin{aligned} \upsilon^0 &= u^0 \\ \upsilon^i &= u^i - \mathrm{v}^i u^0 \qquad (i = 1, 2, 3) \ . \end{aligned} \tag{10}$$

Physically, the 4-velocity υ^m is the 4-velocity of the particle relative to the inflowing river of space. For example, the spatial components υ^i of the 4-velocity are zero if the particle is becalmed in the river, flowing inward with the river. Equations (10) say that if the particle moves a distance $\delta x^\mu = u^\mu \delta\tau$ in the coordinate frame, then it moves a distance $\delta\xi^m = \upsilon^m \delta\tau$ in the tetrad frame

$$\begin{aligned} \delta\xi^0 &= \delta t_{\mathrm{ff}} \\ \delta\xi^i &= \delta x^i - \mathrm{v}^i \delta t_{\mathrm{ff}} \qquad (i = 1, 2, 3) \ . \end{aligned} \tag{11}$$

In words, the spatial distance $\delta\xi^i$ moved relative to the river frame is the distance δx^i in the coordinate frame minus the distance $\mathrm{v}^i \delta t_{\mathrm{ff}}$ moved by the river in proper time $\delta\tau$.

Since the 4-velocity υ^m is relative to a locally inertial frame, the rules of special relativity apply. The 4-velocity squared $\upsilon_m \upsilon^m = \eta_{mn} \upsilon^m \upsilon^n$ is a scalar, and is constant along the path of the particle. For a massive particle

$$\upsilon_m \upsilon^m = -1 \tag{12}$$

while for a massless photon

$$\upsilon_m \upsilon^m = 0 \ . \tag{13}$$

You know how to update the position of the particle from its 4-velocity, but how do you update the 4-velocity? This is determined by the equations of motion, which a general relativistic calculation shows to be

$$\begin{aligned} \frac{\mathrm{d}\upsilon^0}{\mathrm{d}\tau} &= \frac{\partial \mathrm{v}_i}{\partial x^j} \upsilon^j \upsilon^i \\ \frac{\mathrm{d}\upsilon^i}{\mathrm{d}\tau} &= \frac{\partial \mathrm{v}^i}{\partial x^j} \upsilon^j \upsilon^0 \qquad (i = 1, 2, 3) \ . \end{aligned} \tag{14}$$

Equations (14) have the following interpretation. In an interval $\delta\tau$ of proper time, a particle moves a spatial distance $\delta x^i = u^i \delta\tau$. The velocity v^i of the infalling river at the new position differs from the velocity at the old position by $\delta x^j \, \partial \mathrm{v}^i / \partial x^j$. However, in the river model, a particle moving in the river sees not the full change in river velocity relative to the background coordinates, but only the tidal change

$$\delta \mathrm{v}^i = \delta\xi^j \frac{\partial \mathrm{v}^i}{\partial x^j} \qquad (i = 1, 2, 3) \tag{15}$$

in the river velocity relative to the infalling locally inertial river frame. For example, if the particle is becalmed in the river, infalling with it, then $\delta\xi^j = 0$, and the particle sees no change in the river velocity. The infinitesimal tidal change $\delta \mathrm{v}^i$ in the river velocity induces a Lorentz boost in the 4-vector υ^m

$$\begin{aligned} \upsilon^0 &\rightarrow \upsilon^0 + \delta \mathrm{v}_i \, \upsilon^i \\ \upsilon^i &\rightarrow \upsilon^i + \delta \mathrm{v}^i \, \upsilon^0 \qquad (i = 1, 2, 3) \ . \end{aligned} \tag{16}$$

Equations (15) and (16) reproduce the equations of motion (14).

3. Rotating Black Hole

The river model for a rotating black hole is similar to that for a spherical black hole, with two major differences. First, the natural coordinate system is spheroidal rather than spherical. Second, the infalling river of space is characterized not only by a velocity, but also by a twist. That is, the river has a Lorentz structure, characterized by six numbers (velocity and rotation), not just three (velocity). As a test particle moves in free-fall around the black hole, it changes its velocity and rotation in response to tidal changes in the velocity and twist of the river along its path.

For a rotating black hole of mass M, electric charge Q, and angular momentum a per unit mass, the river metric is, in oblate spheroidal coordinates $(t_{\mathrm{ff}}, r, \theta, \phi)$,

$$\mathrm{d}s^2 = -\,\mathrm{d}t_{\mathrm{ff}}^2 + \left[\frac{\rho \mathrm{d}r}{R} + \frac{\mathrm{v}R}{\rho} (\mathrm{d}t_{\mathrm{ff}} - a \sin^2\!\theta \, \mathrm{d}\phi_{\mathrm{ff}}) \right]^2 + \rho^2 \mathrm{d}\theta^2 + R^2 \sin^2\!\theta \, \mathrm{d}\phi_{\mathrm{ff}}^2 \tag{17}$$

where v is the river velocity

$$\mathrm{v} = \frac{(2Mr - Q^2)^{1/2}}{R} \tag{18}$$

and R and ρ are

$$R \equiv (r^2 + a^2)^{1/2} \ , \quad \rho \equiv (r^2 + a^2 \cos^2 \theta)^{1/2} \ . \tag{19}$$

Horizons occur where the river velocity v equals the speed of light

$$\mathrm{v} = \pm 1 \tag{20}$$

with $\mathrm{v} = 1$ for black hole horizons, and $\mathrm{v} = -1$ for white hole horizons. The metric (17) for the Kerr-Newman geometry was discovered by Doran (2000). Oblate spheroidal coordinates (r, θ, ϕ) are not the same as spherical coordinates, but rather are related to Cartesian coordinates (x, y, z) by

$$
\begin{aligned}
x &= R \sin\theta \cos\phi_{\rm ff} \\
y &= R \sin\theta \sin\phi_{\rm ff} \\
z &= r \cos\theta \ .
\end{aligned}
\tag{21}
$$

The spheroidal radial coordinate r is given implicitly in terms of x, y, z by

$$
r^4 - r^2(x^2 + y^2 + z^2 - a^2) - a^2 z^2 = 0 \ . \tag{22}
$$

Rewrite the river metric (17) in Cartesian coordinates $x^\mu = (x^0, x^1, x^2, x^3) = (t_{\rm ff}, x, y, z)$, with the rotation axis along the z-direction:

$$
{\rm d}s^2 = \eta_{\mu\nu}({\rm d}x^\mu - {\rm v}^\mu \alpha_\kappa {\rm d}x^\kappa)({\rm d}x^\nu - {\rm v}^\nu \alpha_\lambda {\rm d}x^\lambda) \ . \tag{23}
$$

Here the components ${\rm v}^\mu$ of the river velocity are

$$
{\rm v}^\mu = \frac{vR}{\rho} \left(0 \, , \, -\frac{xr}{R\rho} \, , \, -\frac{yr}{R\rho} \, , \, -\frac{zR}{r\rho} \right) \tag{24}
$$

and $\alpha_\mu {\rm d}x^\mu = {\rm d}t_{\rm ff} - a \sin^2\!\theta \, {\rm d}\phi_{\rm ff}$ has components

$$
\alpha_\mu = \left(1 \, , \, \frac{ay}{R^2} \, , \, -\frac{ax}{R^2} \, , \, 0 \right) \ . \tag{25}
$$

The azimuthal vector α_μ is related to the 4-velocity of the horizon. Its spatial components point in the (negative) azimuthal direction, in the direction opposite to the rotation of the black hole.

As in the spherical case, introduce a set of locally inertial tetrad frames, and associated locally inertial coordinates $\xi^m \equiv (\xi^0, \xi^1, \xi^2, \xi^3)$, attached to observers who free-fall radially from zero velocity (with zero angular momentum) at infinity. Such freely-falling observers are comoving with the infalling river of space, and have coordinate 4-velocity

$$
u^\mu_{\rm ff} = (1, {\rm v}^1, {\rm v}^2, {\rm v}^3) \ . \tag{26}
$$

The 4-velocity υ^m of a test particle with respect to the locally inertial tetrad frame at the position of the particle is

$$
\upsilon^m \equiv \frac{{\rm d}\xi^m}{{\rm d}\tau} \tag{27}
$$

and is related to the coordinate 4-velocity u^μ by

$$
\begin{aligned}
\upsilon^0 &= u^0 \\
\upsilon^i &= u^i - {\rm v}^i \alpha_\mu u^\mu \qquad (i = 1, 2, 3) \ .
\end{aligned}
\tag{28}
$$

Equations (28) say that if in an interval of proper time $\delta\tau$ the particle moves a coordinate distance $\delta x^\mu = u^\mu \delta\tau$, then relative to the tetrad frame, that is, relative to the locally inertial frame of an observer who is comoving with the infalling river, the particle moves a distance

$$
\delta\xi^m = \delta x^m - {\rm v}^m \alpha_\mu \delta x^\mu \ . \tag{29}
$$

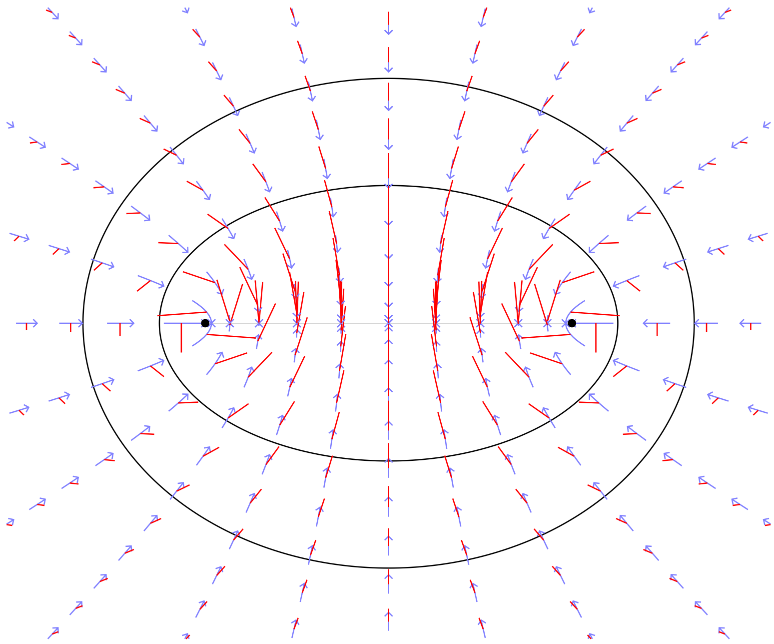

Figure 1: The velocity and twist fields for an uncharged (Kerr) black hole with angular momentum per unit mass $a = 0.95$. The arrowed lines show the magnitude and direction of the river velocity, while the unarrowed lines emerging from the arrowed lines show the magnitude and axis of the river twist. The confocal ellipses show the outer and inner horizons, and the large dots at the foci of the ellipses indicate the ring singularity. In the vacuum Kerr solution, the river velocity goes to zero at the horizontal disc bounded by the ring singularity, then turns around and rebounds through a white hole into a new universe.

One recognizes the right hand side of equation (29) as having the same form as a factor of the Doran-Cartesian metric (23).

The big difference between rotating and spherical black holes is that the river is characterized by a twist as well as a velocity. The velocity and twist are together specified by a single bivector (antisymmetric tensor) river field ω_{kn}

$$\omega_{kn} = \alpha_k \mathrm{v}_n - \alpha_n \mathrm{v}_k + \varepsilon_{0kni}\, \zeta^i \tag{30}$$

where the vector ζ^i is

$$\zeta^i = (0, 0, 0, \zeta) \ , \quad \zeta = a \int_r^\infty \frac{\mathrm{v}\, \mathrm{d}r}{R^2} \tag{31}$$

which points vertically upward along the rotation axis of the black hole. The river field defines a velocity and a rotation, or twist, at each point of the black hole geometry. Components of ω_{kn} in which one of the indices k or n is 0 (time) define a velocity, while components in which both indices k and n are $1, 2, 3$ (space) define a spatial rotation, or twist. The velocity is just the river velocity v_n

$$\omega_{0n} = \mathrm{v}_n \ , \tag{32}$$

while the angle and axis of the river twist are given by the rotation vector

$$\mu^i = \frac{1}{2}\, \varepsilon^{ikn}\, \omega_{kn} = \varepsilon^{ikn}\, \alpha_k \mathrm{v}_n + \zeta^i \qquad (i, k, n = 1, 2, 3) \ . \tag{33}$$

Like the velocity vector v_i, the twist vector μ^i at each point lies in the plane of constant free-fall azimuthal angle ϕ_{ff}, since it is a sum of two vectors $\varepsilon^{ikn}\, \alpha_k \mathrm{v}_n$ and ζ^i both of which are orthogonal to the azimuthal vector α_k.

The equation of motion of an unaccelerated test particle in the river frame is

$$\frac{\mathrm{d}\upsilon^k}{\mathrm{d}\tau} = \eta^{kl} \frac{\partial \omega_{ln}}{\partial x^m} \upsilon^m \upsilon^n \ . \tag{34}$$

The equation of motion (34) can be interpreted as follows. In an infinitesimal interval $\delta\tau$ of proper time, a particle moves a distance $\delta\xi^m = \upsilon^m \delta\tau$ relative to the infalling river of space. As a result of its motion through the river, the particle experiences a tidal change

$$\delta\omega_{kn} = \frac{\partial \omega_{kn}}{\partial x^m} \delta\xi^m \tag{35}$$

in the river field, which generalizes equation (15) for spherical black holes. The tidal change $\delta\omega_{kn}$ in the river field is an infinitesimal Lorentz transformation, and it induces a Lorentz boost and rotation in the 4-vector υ^k

$$\upsilon^k \to \upsilon^k + \delta\omega^k_{\ n}\, \upsilon^n \ . \tag{36}$$

Equations (35) and (36) reproduce the equations of motion (34).