1. Gravitational lensing
In a previous problem you found that, in the weak field limit, light passing a spherical mass $M$ at impact parameter $y$ is deflected by angle
\[ \Delta \phi = \frac{4GM}{y c^2} \tag{1.1} \]

(a) Lensing equation

Argue that the deflection angle $\Delta \phi$ is related to the angles $\alpha$ and $\beta$ illustrated in the lensing diagram above by
\[ \alpha D_S = \beta D_S + \Delta \phi D_{LS} \tag{1.2} \]

Hence or otherwise obtain the “lensing equation” in the form commonly used by astronomers
\[ \beta = \alpha - \frac{\alpha_E^2}{\alpha} \tag{1.3} \]

where
\[ \alpha_E = \left( \frac{4GM}{c^2} \frac{D_{LS}}{D_L D_S} \right)^{1/2} \tag{1.4} \]

(b) Solutions
Equation (1.3) has two solutions for the apparent angles $\alpha$ in terms of $\beta$. What are they? Sketch both solutions on a lensing diagram similar to Figure 1.
Figure 2: The appearance of a lensed source. The lens in this case is a black hole, whose physical size is the filled circle, and whose apparent (lensed) size is the surrounding unfilled circle.

(c) Magnification

Figure 2 illustrates the appearance of a finite-sized source lensed by a gravitational lens. If the source is far from the lens, then the source redshift is unchanged by the gravitational lensing. But the distortion changes the apparent brightness of the source by a magnification $\mu$ equal to the ratio of the apparent area of the lensed source to that of the unlensed source. For a small source, the ratio of areas is

$$\mu = \frac{y_A dy_A}{y_S dy_S}.$$  \hfill (1.5)

What is the magnification of a small source in terms of $\alpha$ and $\alpha_E$? When is the magnification largest?

(d) Einstein ring around the Sun?

The case $\alpha = \alpha_E$ evidently corresponds to the case where the source is exactly behind the lens, $\beta = 0$. In this case the lensed source appears as an “Einstein ring” of light around the lens. Could there be an Einstein ring around the Sun, as seen from Earth?

(e) Einstein ring around Sgr A*

What is the maximum possible angular size of an Einstein ring around the $4 \times 10^6 M_\odot$ black hole at the center of our Milky Way, 8 kpc away? Might this be observable?