1. Equations of motion in weak gravity

Consider the Newtonian metric

\[ ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)(dx^2 + dy^2 + dz^2) \]  

(1.1)

where \( \Phi(x,y,z) \) is the familiar Newtonian gravitational potential, a function only of the spatial coordinates \( x, y, z \), not of time \( t \).

(a) Connection coefficients

Confirm that the non-zero connection coefficients are (coefficients as below but with the last two indices swapped are the same by the no-torsion condition \( \Gamma_{\kappa \mu \nu} = \Gamma_{\nu \mu \kappa} \))

\[ \Gamma_{t\alpha} = \Gamma_{\alpha t} = \Gamma_{\beta \beta \alpha} = -\Gamma_{\beta \alpha \beta} = -\Gamma_{\alpha \alpha \alpha} = -\frac{\partial \Phi}{\partial x^\alpha} \quad (\alpha \neq \beta = x, y, z) \]  

(1.2)

[Hint: Work to linear order in \( \Phi \). You are welcome to use the mathematica notebook metric.nb posted on the website, but if you do, please tell me.]

(b) Energy of a massive particle

Consider a massive, non-relativistic particle moving with 4-velocity \( u^\mu \equiv dx^\mu/d\tau = \{u^t, u^x, u^y, u^z\} \). Show that \( u_\mu u^\mu = -1 \) implies that

\[ u^t = 1 + \frac{1}{2}u^2 - \Phi \]  

(1.3)

whereas

\[ u_t = -\left(1 + \frac{1}{2}u^2 + \Phi\right) \]  

(1.4)

where \( u \equiv [(u^x)^2 + (u^y)^2 + (u^z)^2]^{1/2} \). One of \( u^t \) or \( u_t \) is constant along the geodesic. Which one? [Hint: Work to linear order in \( \Phi \). Note that \( u^2 \) is of linear order in \( \Phi \). As regards which of \( u^t \) or \( u_t \) is constant, notice that the metric is independent of time because \( \Phi(x,y,z) \) is being assumed to be a function only of the spatial coordinates \( x, y, z \), not of time \( t \).]

(c) Equation of motion of a massive particle

From the geodesic equation

\[ \frac{du^\kappa}{d\tau} + \Gamma_{\mu \nu}^{\kappa} u^\mu u^\nu = 0 \]  

(1.5)

show that

\[ \frac{du^\alpha}{dt} = -\frac{\partial \Phi}{\partial x^\alpha} \quad \text{for } \alpha = x, y, z \]  

(1.6)

Why is it legitimate to replace \( d\tau \) by \( dt \)? Show further that

\[ \frac{du^t}{dt} = -2u^\alpha \frac{\partial \Phi}{\partial x^\alpha} \]  

(1.7)
with implicit summation over $\alpha = x, y, z$. Does the result agree with what you’d expect from equation (1.3)? [Hint: A consistent perturbative approach is to keep only the lowest order non-vanishing parts of an equation, discarding the higher order parts as negligible.]

(d) Energy of a massless particle

For a massless particle, the proper time along a geodesic is zero, and the affine parameter $\lambda$ must be used instead of the proper time. The 4-velocity of a massless particle can be defined to be (and really this is just the 4-momentum up to an arbitrary overall factor) 

$$v^\mu \equiv dx^\mu/d\lambda = \{v^t, v^x, v^y, v^z\}. $$

Show that $v_\mu v^\mu = 0$ implies that

$$v^t = (1 - 2\Phi) v, \quad (1.8)$$

whereas

$$v_t = -v, \quad (1.9)$$

where $v \equiv [(v^x)^2 + (v^y)^2 + (v^z)^2]^{1/2}$. One of $v^t$ or $v_t$ is constant. Which one?

(e) Equation of motion of a massless particle

From the geodesic equation

$$\frac{dv^\kappa}{d\lambda} + \Gamma^\kappa_{\mu\nu} v^\mu v^\nu = 0 \quad (1.10)$$

show that the spatial components $v \equiv \{v^x, v^y, v^z\}$ satisfy

$$\frac{dv}{d\lambda} = 2 v \times (v \times \nabla \Phi) \quad (1.11)$$

where boldface symbols represent 3D vectors, and in particular $\nabla \Phi$ is the spatial 3D gradient $\nabla \Phi \equiv \partial \Phi/\partial x^\alpha = \{\partial \Phi/\partial x, \partial \Phi/\partial y, \partial \Phi/\partial z\}$. [Hint: Recall that the 3D vector triple product satisfies $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$.]

(f) Interpret

Interpret your answer, equation (1.11). In what ways does this equation for the acceleration of photons differ from the equation governing the acceleration of massive particles? [Hint: Without loss of generality, the affine parameter can be normalized so that the photon speed is one, $v = 1$, so that $v$ is a unit vector representing the direction of the photon.]
2. Deflection of light by the Sun

(a) Born approximation

Consider light that passes by a spherical mass \( M \) sufficiently far away that the potential \( \Phi \) is always weak. The potential at distance \( r \) from the spherical mass can be approximated by the Newtonian potential

\[
\Phi = -\frac{GM}{r}.
\]  

(2.1)

Approximate the unperturbed path of light past the mass as a straight line. The plan is to calculate the deflection as a perturbation to the straight line (physicists call this the Born approximation). For definiteness, take the light to be moving in the \( x \)-direction, offset by a constant amount \( y \) away from the mass in the \( y \)-direction (so \( y \) is the impact parameter, or periapsis). Argue that equation (1.11) becomes

\[
\frac{dv_y}{d\lambda} = v^x \frac{dv_y}{dx} = -2 (v^x)^2 \frac{\partial \Phi}{\partial y}.
\]

(2.2)

Integrate this equation to show that

\[
\Delta v_y v^x = -\frac{4GM}{y}.
\]

(2.3)

Argue that this equals the deflection angle \( \Delta \phi \).

(b) Deflection of light by the Sun

Calculate the predicted deflection angle \( \Delta \phi \) in arcseconds for light that just grazes the limb of the Sun.