1. Earth metric
The metric just above the surface of the Earth is well-approximated by
\[ ds^2 = - (1 + 2\Phi) dt^2 + (1 - 2\Phi) dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) , \]
where
\[ \Phi(r) = - \frac{GM}{r} \]
is the familiar Newtonian gravitational potential.

(a) Proper time
Consider an object at fixed radius \( r \), moving along the equator \( \theta = \pi/2 \) with constant non-relativistic velocity \( r \frac{d\phi}{dt} = v \). Compare the proper time of this object with that at rest at infinity. [Hint: Work to first order in the potential \( \Phi \). Regard \( v^2 \) as first order in \( \Phi \). Why is that reasonable?]

(b) Orbits
Consider a satellite in orbit about the Earth. The conservation of energy \( E \) per unit mass, angular momentum \( L \) per unit mass, and rest mass per unit mass are expressed by
\[ u_t = -E , \quad u_\phi = L , \quad u_\mu u^\mu = -1 . \]
For equatorial orbits, \( \theta = \pi/2 \), show that the radial component \( u^r \) of the 4-velocity satisfies
\[ u^r = \sqrt{2(\Delta E - U)} , \]
where \( \Delta E \) is the energy per unit mass of the particle excluding its rest mass energy,
\[ \Delta E = E - 1 , \]
and the effective potential \( U \) is
\[ U = \Phi + \frac{L^2}{2r^2} . \]
[Hint: Neglect air resistance. Remember to work to first order in \( \Phi \). Treat \( \Delta E \) and \( L^2 \) as first order in \( \Phi \). Why is that reasonable?]

(c) Circular orbits
From the condition that the potential \( U \) be an extremum, find the circular orbital velocity \( v = r \frac{d\phi}{dt} \) of a satellite at radius \( r \).

(d) Special and general relativistic corrections for satellites
Compare the proper time of a satellite in circular orbit to that of a person at rest at infinity. Express your answer in the form
\[ \frac{d\tau_{\text{satellite}}}{dt} - 1 = -\Phi_\oplus (f_{\text{GR}} + f_{\text{SR}}) , \]
where $f_{GR}$ and $f_{SR}$ are the general relativistic and special relativistic corrections, and $\Phi_\oplus$ is the dimensionless gravitational potential at the surface of the Earth,

$$\Phi_\oplus = -\frac{GM_\oplus}{c^2 R_\oplus}.$$  \hspace{1cm} (1.8)

What is the value of $\Phi_\oplus$ in milliseconds per year?

**(e) Special and general relativistic corrections for satellites vs. Earth observer**

Compare the proper time of a satellite in circular orbit to that of a person on Earth at one of the poles (so the person has no motion from the Earth’s rotation). Express your answer in the form

$$\frac{d\tau_{\text{satellite}}}{dt} - \frac{d\tau_{\text{person}}}{dt} = -\Phi_\oplus (f_{GR} + f_{SR}) .$$  \hspace{1cm} (1.9)

At what satellite radius $r$, in units of Earth radius $R_\oplus$, do the special and general relativistic corrections cancel?

**(f) Special and general relativistic corrections for ISS and GPS satellites**

What are the corrections (be careful to get the sign right!) in units of $\Phi_\oplus$, and in units of ms yr$^{-1}$, for (i) a satellite in low Earth orbit, such as the International Space Station; (ii) a nearly geostationary satellite, such as a GPS satellite? Google the numbers that you may need.