The main-sequence: homologous stars

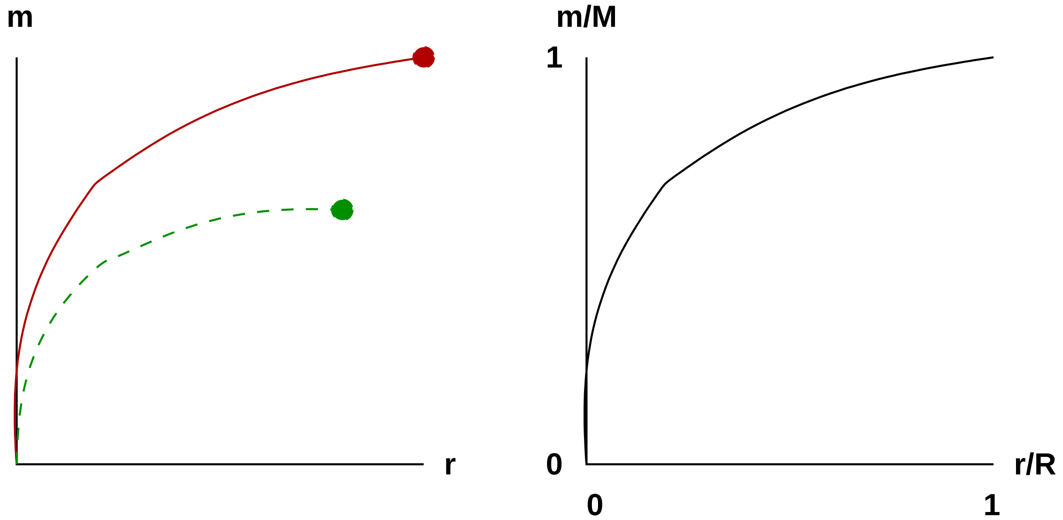
For stars on the main-sequence, possible to derive some scaling properties *without* solving full set of structure equations.

Basic idea is that it is easier (more accurate) to analytically estimate the *relative* properties of stars of different masses than the absolute properties of any one star.

Consider a *reference star* with mass M_0 and radius R_0. If, at any point (r, m) within a second star of mass M, radius R,

$$\begin{aligned} r &= \frac{R}{R_0} r_0 \\ m &= \frac{M}{M_0} m_0 \end{aligned}$$

then the two stars are said to be **homologous** to one another. Physically, this means that the stars have the same *relative* interior mass distribution:



Write the hydrostatic equilibrium equation,

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

...in Lagrangian form using the continuity equation.

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\rightarrow \ \frac{dP}{dm} = -\frac{Gm}{4\pi r^4}.$$

Dimensionally,

$$P \propto \frac{M^2}{R^4}.$$

Write the pressure, opacity, and energy generation rate as power laws,

$$\begin{aligned} P &= P_0 \rho^{\chi_\rho} T^{\chi_T} \\ \kappa &= \kappa_0 \rho^n T^{-s} \\ \epsilon &= \epsilon_0 \rho^\lambda T^\nu. \end{aligned}$$

We have already determined these exponents for different physical processes.

Taking logarithmic derivatives of the power-law expression for the pressure gives,

$$d \ln P = \chi_\rho d \ln \rho + \chi_T d \ln T.$$

Eliminating the $d \ln P$ term using the scaling for hydrostatic equilibrium we obtain,

$$4d \ln R + \chi_\rho d \ln \rho + \chi_T d \ln T = 2d \ln M.$$

Now, assume that the quantities R, ρ, T and L can all be written as power-laws in the mass M,

$$\begin{aligned} R &\propto M^{\alpha_R} \\ \rho &\propto M^{\alpha_\rho} \\ T &\propto M^{\alpha_T} \\ L &\propto M^{\alpha_L}. \end{aligned}$$

Substituting into the previous expression gives,

$$4\alpha_R + \chi_\rho \alpha_\rho + \chi_T \alpha_T = 2,$$

which is one relation between the α's. Repeating the exercise for the mass, energy generation and energy transport equations yields 4 equations $\rightarrow$ enough to specify a solution for the α's.

Possible source of confusion: in these equations the α's are the unknowns. Assume we know the values of χ_ρ, ν, n appropriate for the kind of star we are considering.

Equations for the α's can be written in matrix form as,

$$\begin{pmatrix} 3 & 1 & 0 & 0 \\ 4 & \chi_\rho & 0 & \chi_T \\ 0 & \lambda & -1 & \nu \\ 4 & -n & -1 & 4+s \end{pmatrix} \begin{pmatrix} \alpha_R \\ \alpha_\rho \\ \alpha_L \\ \alpha_T \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$$

where we have assumed that energy is transported via radiative diffusion rather than convection.

The determinant of the matrix on the LHS is,

$$D_{\mathrm{rad}} = (3\chi_\rho - 4)(\nu - s - 4) - \chi_T(3\lambda + 3n + 4).$$

Assuming that $D_{\mathrm{rad}} \neq 0$, the solutions for the power-law scalings of radius, density, luminosity and temperature with mass are,

$$\begin{aligned}
\alpha_R &= \frac{1}{3}\left[1 - \frac{2\left(\chi_T + \nu - s - 4\right)}{D_{\mathrm{rad}}}\right] \\
\alpha_\rho &= \frac{2\left(\chi_T + \nu - s - 4\right)}{D_{\mathrm{rad}}} \\
\alpha_L &= 1 + \frac{2\lambda\left(\chi_T + \nu - s - 4\right) - 2\nu\left(\chi_\rho + \lambda + n\right)}{D_{\mathrm{rad}}} \\
\alpha_T &= -\frac{2\left(\chi_\rho + \lambda + n\right)}{D_{\mathrm{rad}}}
\end{aligned}$$

Similarly opaque expressions can be derived for the case of convective energy transport.

Already noted that for stars with deep surface convection zones, boundary conditions at the surface have a major influence on the structure. Therefore, apply analysis to stars with $M \geq M_\odot$.

On the upper main sequence, expect,

- Opacity in the central region given by electron scattering,

$$n = s = 0.$$

- CNO cycle most important during hydrogen burning,

$$\begin{aligned} \lambda &= 1 \\ \nu &= 15. \end{aligned}$$

- Assume ideal gas pressure,

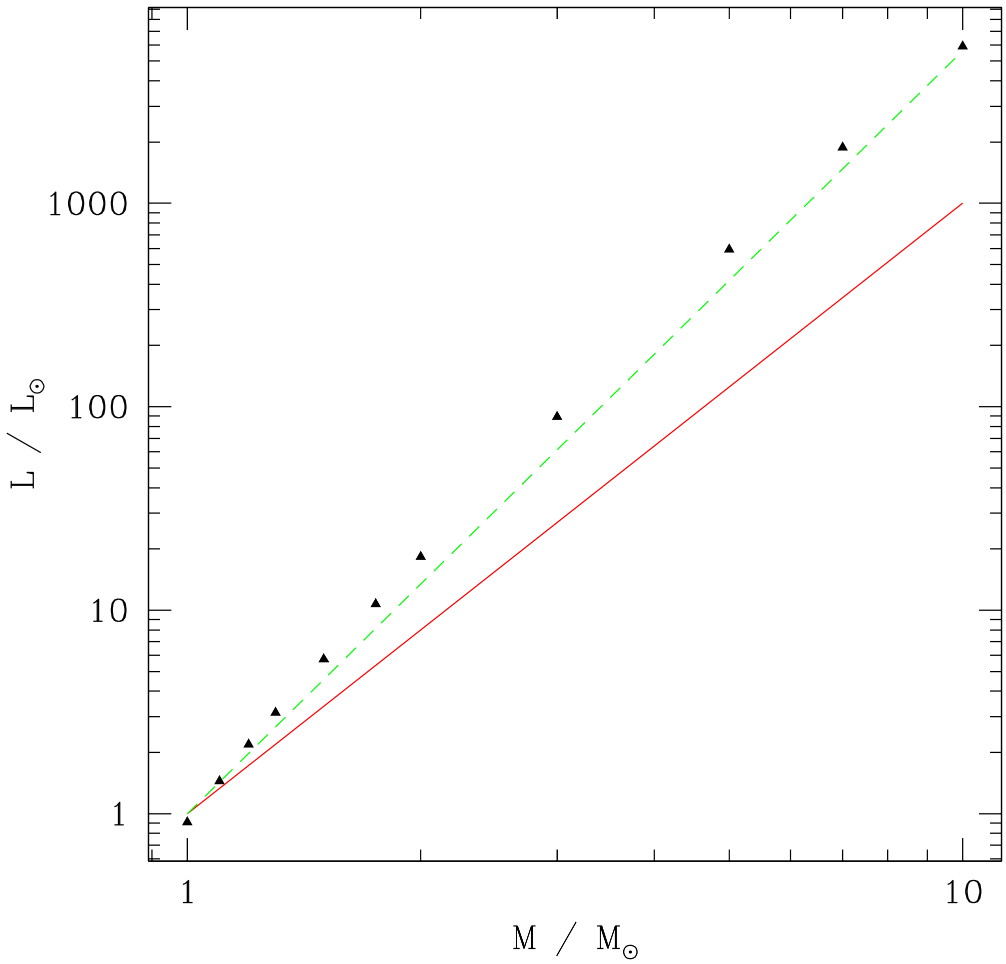
$$\chi_\rho = \chi_T = 1.$$

Find,

$$\begin{aligned} \frac{R}{R_\odot} &= \left(\frac{M}{M_\odot}\right)^{0.78} \\ \frac{L}{L_\odot} &= \left(\frac{M}{M_\odot}\right)^{3.0}. \end{aligned}$$

Observed values quoted in *Hansen & Kawaler* are 0.75 and 3.5, so the dimensional analysis is reasonably accurate.

Plotting the luminosity as a function of mass for models computed using ZAMS:



Slope of 3 is not too bad, but 3.75 provides a better fit.