Nuclear reactions: resonances

Consider a generic nuclear reaction in which a nucleus X reacts with particle a to form nucleus Y with emission of a particle b,

$$X + a \rightarrow Y + b.$$

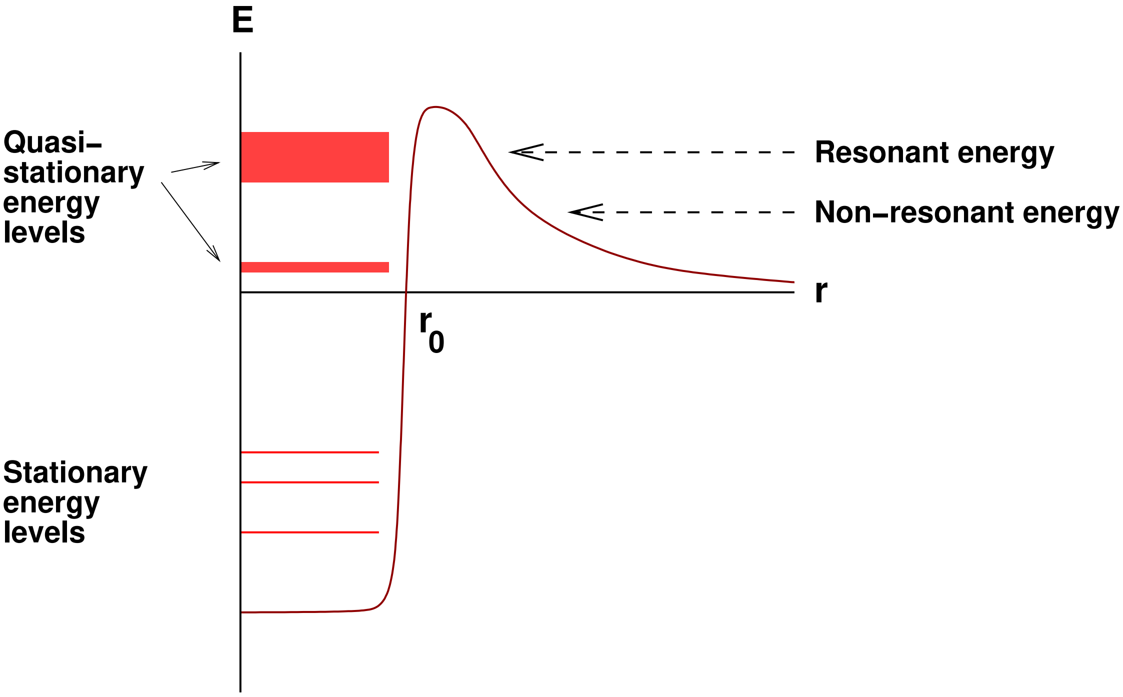
Can imagine this occurring in several steps:

(i) Tunnelling through the Coulomb barrier.

(ii) Formation of an excited compound nucleus C^*, whose energy will depend upon the reaction and on the kinetic energy of the reacting particles.

(iii) Decay of the compound nucleus via the emission of photons, neutrons, protons, α particles, electrons etc. Since timescale for β decay is $\sim$ s, negligible probability if other reactions can occur.

Cross-section for some astrophysically interesting reactions depends critically upon energy level structure of C^* – at resonant energies σ can be boosted by orders of magnitude.



Compound nucleus has,

- Stationary energy levels, corresponding to excited atomic states, which can decay via γ emission.
- Quasi-stationary levels, which can decay via particles tunnelling back through the Coulomb barrier. Lifetime of these states is shorter $\rightarrow$ they are also broader, with width,

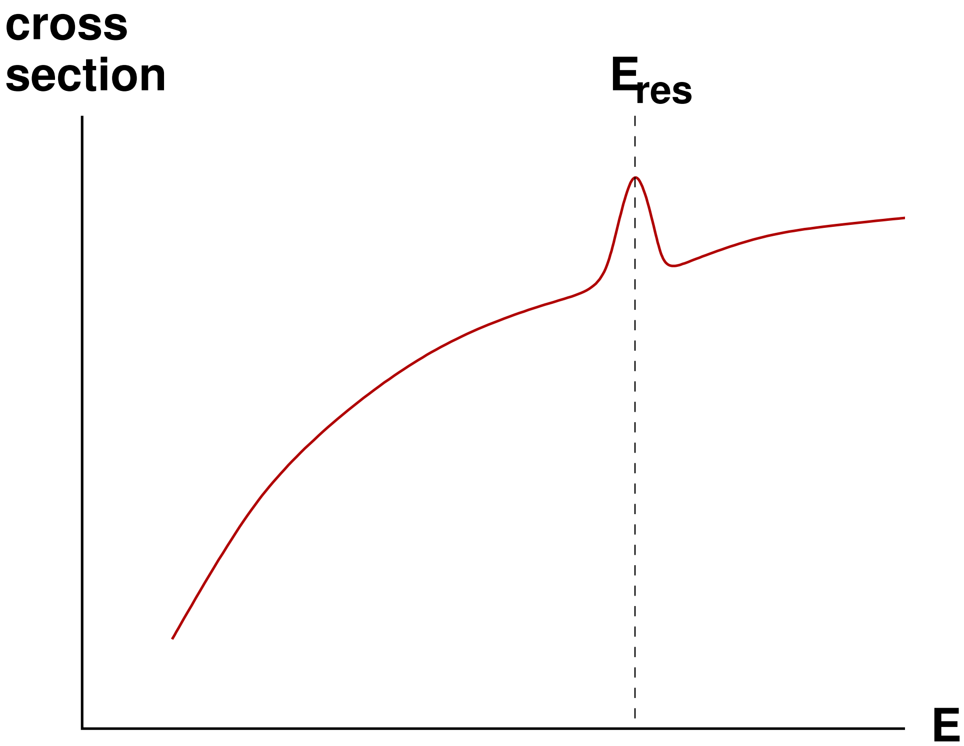
  $$\Gamma = \frac{\hbar}{\tau},$$

  where τ is the lifetime. Γ will increase for higher energy states. At high enough energies, neighbouring levels may overlap, leading to a continuum of excited energy states.

Energy dependence of the cross-section has the form,

$$\xi(E) \propto \frac{1}{(E - E_{\rm res})^2 + (\Gamma/2)^2},$$

near a resonance at energy $E = E_{\rm res}$.



Maximum cross-section will be of the order of the geometric cross-section $\pi\times$ (de Broglie wavelength)^2 which is $\propto E^{-1}$. Combining this with the dominant exponential tunnelling factor, write,

$$\sigma(E) = SE^{-1}e^{-2\pi\eta}.$$

S is the *astrophysical cross-section factor*. Away from resonances, it will generally vary slowly with energy for a given reaction.

Reaction rates

Aim to determine:

- Which energies of particle contribute most to the total reaction rate (high E gives higher P_0 for fusion, but fewer potential reactions).
- Why reactions have high T dependence.

Follow the description and notation in *Kippenhann & Weigert.*

Consider particles of type j moving with velocity v relative to particles of type k. If the number densities per unit volume are n_j, n_k, then number of reactions per unit time per unit volume is,

$$\tilde{r}_{jk} = n_j n_k \sigma v.$$

If $j = k$ then we must avoid double counting possible pairs of reacting particles,

$$\tilde{r}_{jk} = \frac{1}{1 + \delta_{jk}} n_j n_k \sigma v.$$

Assuming both species of particle have Maxwell-Boltzmann velocity distributions, then the relative velocity v is Maxwellian. Let,

$$E = \frac{1}{2} m v^2$$

where m is the reduced mass,

$$m = \frac{m_j m_k}{m_j + m_k}.$$

The fraction of all pairs with E between E and $E + dE$ is,

$$f(E)dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} e^{-E/kT} dE.$$

The total reaction rate (per unit volume per unit time) is the mono-energetic rate integrated over all energies with an $f(E)$ weighting factor,

$$r_{jk} = \frac{1}{1 + \delta_{jk}} n_j n_k \langle \sigma v \rangle,$$

where,

$$\langle \sigma v \rangle = \int_0^\infty \sigma(E) v f(E) dE.$$

The density and the mass fractions of different elements are the quantities we're most interested in. Thus replace the number densities by mass fractions via,

$$X_i \rho = n_i m_i$$

where m_i is the mass of particles that have mass fraction X_i. If each reaction releases an amount of energy Q, then,

$$\epsilon_{jk} = \frac{1}{1 + \delta_{jk}} \frac{Q}{m_j m_k} \rho X_j X_k \langle \sigma v \rangle.$$

This is the energy generation rate in units of energy per unit mass per unit time.

All the temperature dependence is contained in the factor $\langle \sigma v \rangle$. Assembling previous results,

$$\langle \sigma v \rangle = \frac{2^{3/2}}{(m\pi)^{1/2}} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) e^{-E/kT - \bar{\eta}/E^{1/2}} dE,$$

where,

$$\bar{\eta} = 2\pi\eta E^{1/2} = \pi(2m)^{1/2} \frac{Z_j Z_k e^2}{\hbar}.$$

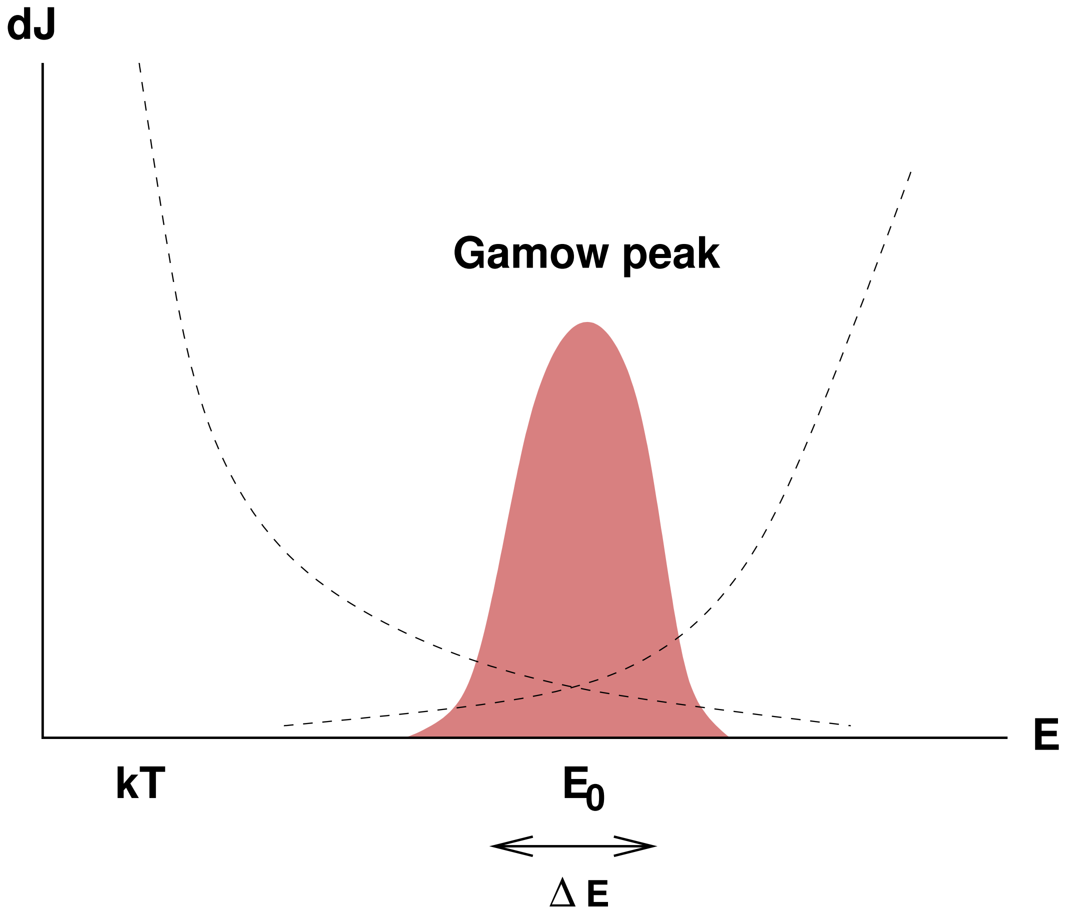
Temporarily ignore the $S(E)$ term. Remaining integral is of the form,

$$J = \int_0^\infty e^{f(E)} dE$$

with,

$$f(E) = -\frac{E}{kT} - \frac{\bar{\eta}}{E^{1/2}}.$$

The integrand is the product of a rapidly rising function (arising from the tunnelling probability) and a rapidly decreasing one (from the Maxwellian).



The integrand peaks at E_0, where $df(E)/dE = 0$. Find,

$$E_0 = \left(\frac{1}{2}\bar{\eta}kT\right)^{2/3}.$$

Since only a small range of energies contributes significantly to the integral, OK for *non-resonant* reactions to assume $S(E) \simeq S_0$, and take the 'constant' S_0 outside the integral. Hence, only need to evaluate J.

To detemine J, define new quantity τ,

$$\tau = 3\frac{E_0}{kT}$$

and expand $f(E)$ about the maximum at E_0 using a series expansion truncated at the quadratic term,

$$\begin{aligned} f(E) &= f(E_0) + f'(E_0)(E - E_0) + \frac{1}{2}f''(E_0)(E - E_0)^2 + ... \\ &= -\tau - \frac{1}{4}\tau\left(\frac{E}{E_0} - 1\right)^2 + ... \end{aligned}$$

With this approximation,

$$J = \int_0^\infty \exp\left[-\tau - \frac{\tau}{4}\left(\frac{E}{E_0} - 1\right)^2\right] dE$$

which evaluates (again approximately) to,

$$J \approx \frac{2}{3}kT\pi^{1/2}\tau^{1/2}e^{-\tau}$$

Final result for non-resonant reactions is,

$$\langle\sigma v\rangle = \frac{4}{3}\left(\frac{2}{m}\right)^{1/2}\frac{1}{(kT)^{1/2}}S_0\tau^{1/2}e^{-\tau} \propto S_0\tau^2e^{-\tau}.$$

Results

(1) Energy at the Gamov peak

Define,

$$W = Z_j^2 Z_k^2 \frac{A_j A_k}{A_j + A_k},$$

such that $W \sim 1$ for the lightest nuclei and will be large for heavier nuclei. Then,

$$\frac{E_0}{kT} \simeq 6.6 W^{1/3} \left(\frac{T}{10^7 \text{ K}} \right)^{-1/3}.$$

E_0 increases with W, so expect separated phases of nuclear burning.

(2) Temperature dependence

Define,

$$\nu = \frac{\partial \ln \langle \sigma v \rangle}{\partial \ln T},$$

as a measure of the temperature sensitivity of the reaction rate. Find,

$$\nu = \frac{\tau}{3} - \frac{2}{3} = 6.6 W^{1/3} \left(\frac{T}{10^7 \text{ K}} \right)^{-1/3} - \frac{2}{3}.$$

$\nu \approx 5$ for the lightest elements, rising to ~ 20 for heavier nuclei. Strong temperature dependence means stars must (almost always) be stable to fluctuations – increased ϵ must lower T.