Convection: mixing length theory

Once convection sets in, there is no reliable analytic way to predict the flux carried due to convection. Lab experiments are of limited utility:

- Incompressible fluids i.e. no scale height.
- Too viscous. Measured via the Reynolds number:

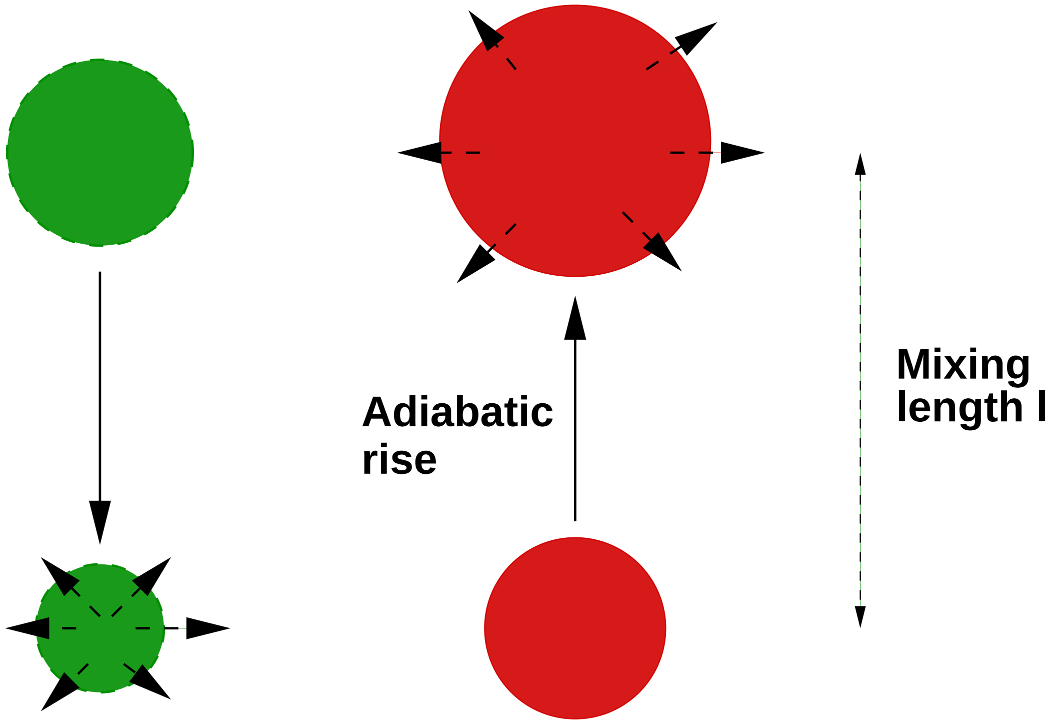
  $$\mathrm{Re} = \frac{LV}{\nu},$$

  where L and V are characteristic lengthscales and velocities, and ν is the viscosity. Taking $L \sim R$, and $V \sim c_s$, obvious that Re will be very large for a star.
- Presence of boundaries.

Thus, use an empirical theory for spherically symmetric stellar models. Mixing length theory has a single adjustable parameter, the *mixing length* l.

Empirical theories have many variants – follow *Clayton's* description here. You may want to look at the more elaborate version in *Hansen & Kawaler*.

Schematically:



- Each mass element rises or falls a distance l adiabatically.
- After one mixing length, the element thermalizes with the local environment.

One might guess that l should be comparable to (or at least, scale with) the local pressure scale height,

$$\lambda_P = \frac{P}{\left|\frac{dP}{dr}\right|}.$$

Thus, write

$$l = \alpha \lambda_P$$

where α is now the adjustable parameter.

After rising adiabatically by distance l, mass element will be hotter than surrounding by an amount,

$$\Delta T = \left(|\frac{dT}{dr}| - |\frac{dT}{dr}|_{ad} \right) l \equiv l \Delta \nabla T,$$

where we define $\Delta \nabla T$ to be the excess of the absolute magnitude of the temperature gradient over the absolute magnitude of the adiabatic temperature gradient.

Note:

- The adiabatic temperature gradient is known.
- The excess of the true gradient over the adiabatic gradient is not known – this is what we want to determine.

If the mass element now thermalizes at constant P, the amount of heat per unit mass released is,

$$\Delta Q = c_P \Delta T = c_P l \Delta \nabla T.$$

If the average velocity of the adiabatic cells is $\bar{V}$, the average excess heat flux is,

$$H = \rho \bar{V} \Delta Q = \rho \bar{V} c_P l \Delta \nabla T.$$

Remaining step is to estimate what value to take for $\bar{V}$.

Mass element is accelerated due to buoyancy – i.e. due to density defecit compared to surroundings. This is given by,

$$\Delta\rho = \left(|\frac{d\rho}{dr}| - |\frac{d\rho}{dr}|_{ad} \right) l \equiv l \Delta\nabla\rho.$$

Average density defecit in the rising element is,

$$\overline{\Delta\rho} = \frac{1}{2} l \Delta\nabla\rho,$$

so the average buoyant force per unit volume is,

$$\bar{F} = g\overline{\Delta\rho} = \frac{1}{2} g l \Delta\nabla\rho$$

where $g = Gm/r^2$ is the local gravity. Resulting acceleration is,

$$a = \frac{\bar{F}}{\rho} = \frac{gl}{2\rho}\Delta\nabla\rho.$$

Acting over distance l, final velocity (assuming a constant acceleration) would be $\sqrt{2al}$. Assume the characteristic velocity is half this,

$$\overline{V} = \frac{1}{2}\left(\frac{GM}{\rho r^2}\Delta\nabla\rho\right)^{1/2} l.$$

Note: even in this simple version there are several places where the numerical factors could be chosen differently.

As we showed last time (when converting the stability criteria from density to temperature), for an ideal nondegenerate gas,

$$\Delta \nabla \rho = \frac{\rho}{T} \Delta \nabla T.$$

Final result is that the heat flux due to convection is given (approximately) by,

$$H = c_P \rho \left(\frac{Gm}{Tr^2} \right)^{1/2} (\Delta \nabla T)^{3/2} \frac{l^2}{2}$$

i.e. scaling with the square of the mixing length and as the excess temperature gradient to the 3/2 power (not linearly because the velocity of the rising elements increases with the excess gradient).

Note: this is a **local** theory for the convective flux – can compute H given only quantities at a single point in the star.

How large an excess gradient is needed?

Easy to show that in the center of a star, a very small excess gradient suffices to carry all the luminosity.

e.g. suppose (notionally) that at $m/M = 0.5$ all the Solar flux was carried by convection. Output from the *Hansen & Kawaler* ZAMS code gives $\lambda_p = 5 \times 10^9$ cm at this radius, along with values for T, ρ etc.

Substituting appropriate values into,

$$(\Delta \nabla T)^{3/2} = \frac{L_r/(4\pi r^2)}{c_P \rho \left(\frac{Gm}{Tr^2}\right)^{1/2} \frac{l^2}{2}},$$

and taking $l = \lambda_P$, obtain,

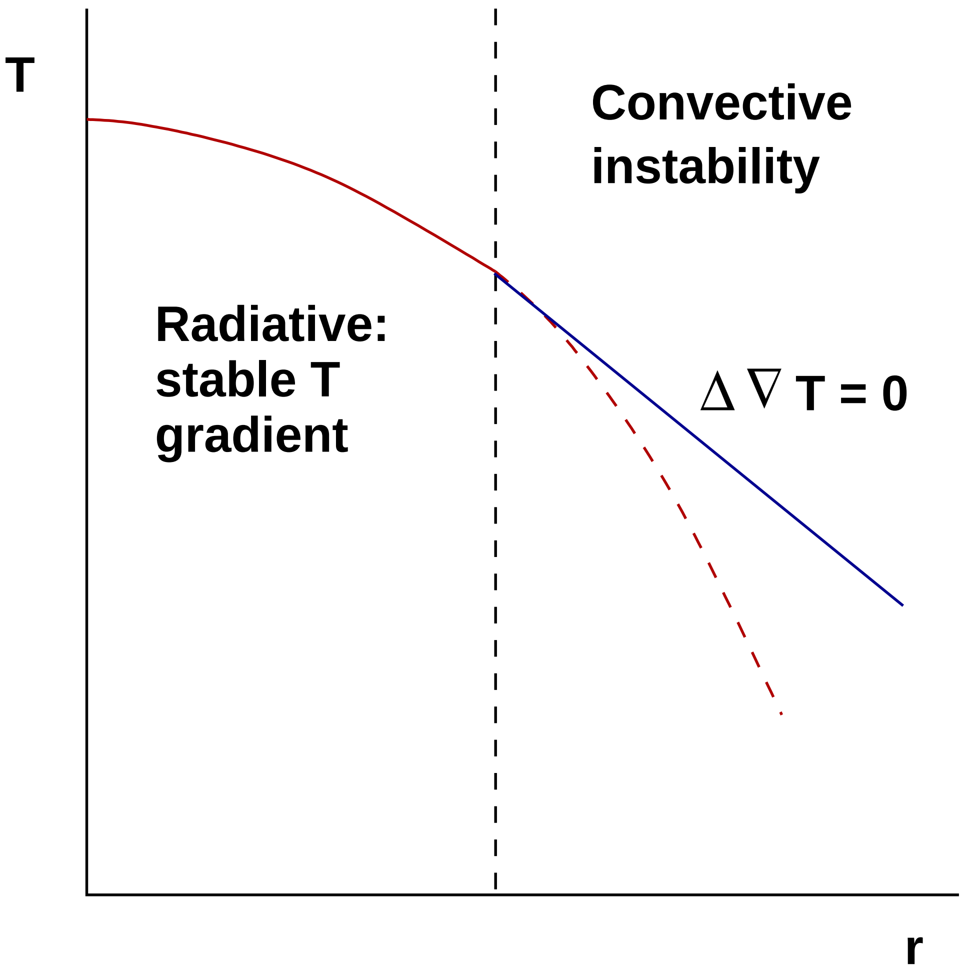
$$\Delta \nabla T \sim 10^{-10} \text{ K cm}^{-1}.$$

Compare to the average temperature gradient in the Sun,

$$\left|\frac{dT}{dr}\right| \approx \frac{T_c}{R} \sim 10^{-4} \text{ K cm}^{-1}.$$

Conclude: the required excess is only $\sim 10^{-6}$ of the temperature gradient.

$\rightarrow$ **When convection occurs in stellar interiors, the resulting temperature gradient equals the adiabatic gradient to high accuracy.**



Simplest approach to convection:

- Compare radiative temperature gradient with the adiabatic gradient,

$$\frac{dT}{dr} = \left(1 - \frac{1}{\Gamma_2}\right) \frac{T}{P} \frac{dP}{dr}.$$

- If gradient is convectively unstable, then resulting gradient is driven to adiabatic gradient to a good approximation.

Complications

In a medium of uniform composition, criteria for stability can be written as,

$$\nabla_{rad} < \nabla_{ad}$$

where,

$$\nabla_{rad} \equiv \left(\frac{d\ln T}{d\ln P}\right)_{rad}.$$

(note that T increases with P, hence the sign flip compared to our previous expression involving r).

This must be modified if the composition changes with radius. Consider a star in which the mean weight μ decreases outwards. Displacement of an element upwards will carry heavier material into lighter surroundings – this will lead to greater stability than the uniform case.

Stability is given by the *Ledoux* criterion,

$$\nabla_{rad} < \nabla_{ad} + \frac{\varphi}{\delta}\nabla_{\mu}$$

where,

$$\delta = -\left(\frac{\partial \ln \rho}{\partial \ln T}\right), \quad \varphi = \left(\frac{\partial \ln \rho}{\partial \ln \mu}\right), \quad \nabla_{\mu} = \left(\frac{d\ln \mu}{d\ln P}\right).$$

Since nuclear reactions increase μ in the inner regions, this is almost always a stabilizing effect.

When convection is efficient, we don't really need to worry about the exact value of the mixing length.

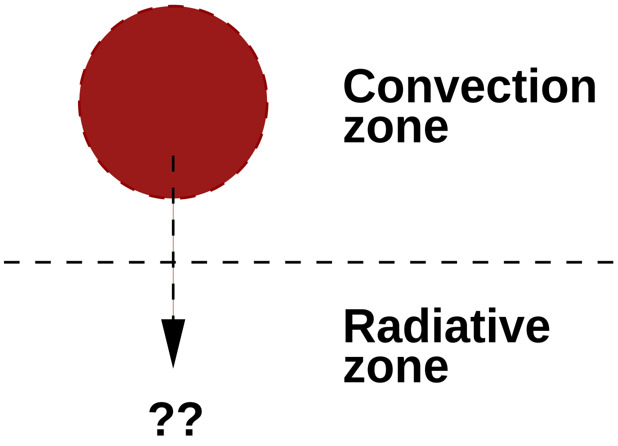
Near the surface of the star, however, λ_p becomes small. A non-negligible excess gradient $\Delta \nabla T$ is required before convection can carry the flux:

- Implies that the actual temperature gradient is intermediate between the adiabatic and radiative gradients – called *superadiabatic* convection.
- Means that the stellar radius, effective temperature etc do depend upon the α of mixing length theory.
- Using the ZAMS code provided with *Hansen & Kawaler*, a factor two change in α leads to a few hundred K difference in T_e.

Schaller et al., A&A Supplement, 96, 269 (1992) quote $\alpha = 1.6 \pm 0.1$. Similar values are quoted for red giants.

Overshooting

Do convective elements overshoot into stably stratified regions due to their inertia?



Naively, expect no – acceleration is small in convection zone but braking is large once elements reach radiative zone.

This is a non-local effect $\rightarrow$ cannot be treated within mixing-length theory adequately. Numerical simulations find overshooting can extend to a significant fraction of λ_p.

'Full spectrum of turbulence'

Refers to an alternative prescription for convection based on the (reasonable) observation that turbulent processes involve a range of lengthscales.

Still approximate, and unclear if this provides a better match to observations than mixing-length approach.

Reference: Canuto & Mazzitelli, ApJ, 370, 295 (1991).