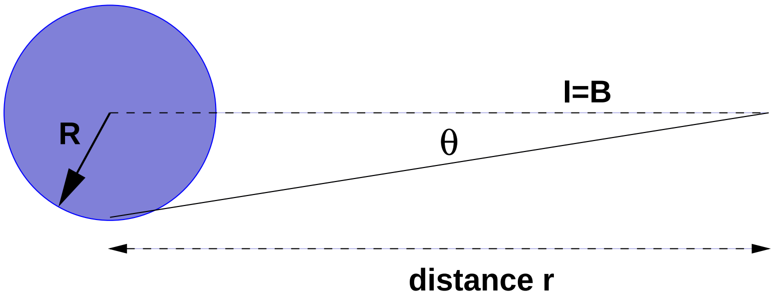
Equation of radiative diffusion



Consider a sphere of uniform brightness B. At an exterior point,

- $I = B$ if the ray intersects the sphere
- $I = 0$ otherwise.

Integrate over the visible area of the sphere to find the flux,

$$F = \int I \cos\theta d\Omega = B \int_0^{2\pi} d\phi \int_0^{\theta_c} \sin\theta \cos\theta d\theta$$

where the upper limit on the θ integral is where a ray just grazes the sphere, $\sin\theta_c = R/r$. The integral gives,

$$\begin{aligned} F &= \pi B(1 - \cos^2\theta_c) = \pi B \sin^2\theta_c \\ F &= \pi B \left(\frac{R}{r}\right)^2 \end{aligned}$$

ie it all works. The specific intensity is constant but the solid angle drops with radius to give the inverse square law.

Setting $r = R$, the flux at the surface of an object of uniform brightness B is $F = \pi B$.

Formal solution of the transfer equation

Start with,

$$\frac{dI_\nu}{d\tau_\nu} + I_\nu = S_\nu$$

and multiply by the integrating factor $e^{\int d\tau_\nu}$.

$$\begin{aligned} e^{\tau_\nu}\frac{dI_\nu}{d\tau_\nu} + e^{\tau_\nu} I_\nu &= e^{\tau_\nu} S_\nu \\ \frac{d}{d\tau_\nu}\left[I_\nu e^{\tau_\nu}\right] &= e^{\tau_\nu} S_\nu \\ I_\nu e^{\tau_\nu} &= \int_0^{\tau_\nu} e^{\tau'_\nu} S_\nu d\tau'_\nu + \text{constant} \end{aligned}$$

When $\tau_\nu = 0$, $I_\nu = I_\nu(0)$. Thus,

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + \int_0^{\tau_\nu} e^{-(\tau_\nu - \tau'_\nu)} S_\nu(\tau'_\nu) d\tau'_\nu$$

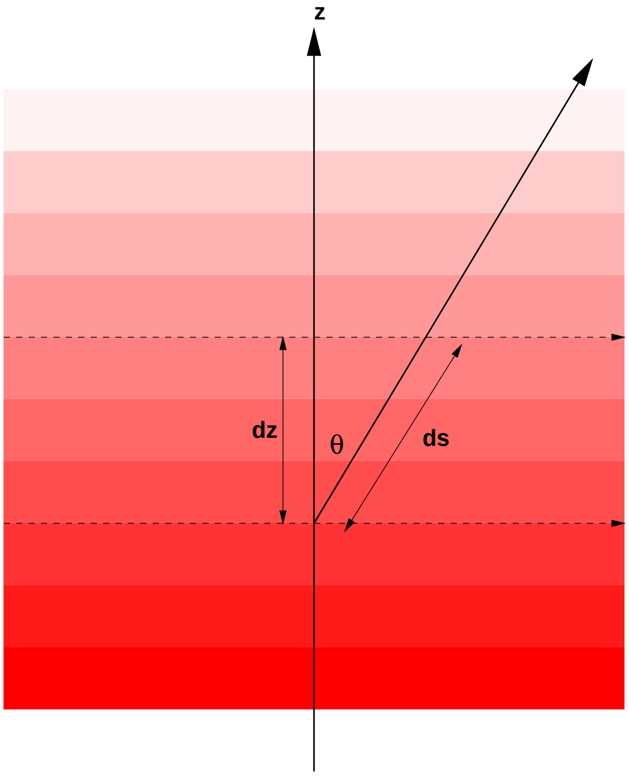
Note: $\tau_\nu = 1$ implies absorption by a factor of e. Hence: the final intensity is the initial intensity diminished by absorption, plus the integrated source function also diminished by absorption.

For a constant source function,

$$I_\nu(\tau_\nu) = I_\nu(0)e^{-\tau_\nu} + S_\nu(1 - e^{-\tau_\nu})$$

and as $\tau_\nu \to \infty$, $I_\nu \to S_\nu$. True generally, the specific intensity approaches the source function at large optical depth.

At high optical depths, $S_\nu \to B_\nu$. However, within a star there is a net radial flux. Hence, there must be some departure from isotropy. Aim to relate the flux to the local temperature gradient.



Geometry for the derivation:

- Plane-parallel medium (all quantities are $f(z)$ only).
- Angular dependence of I_ν only via θ.
- Use variable $\mu = \cos\theta$. Geometry gives,

$$ds = \frac{dz}{\cos\theta} = \frac{dz}{\mu}$$

Using μ and z as variables, the transfer equation,

$$\frac{dI_\nu}{ds} = -\rho\kappa_\nu(I_\nu - S_\nu)$$

becomes,

$$\mu\frac{\partial I_\nu(z,\mu)}{\partial z} = -\rho\kappa_\nu(I_\nu - S_\nu)$$

(note now a partial derivative).

Rearrange this to get,

$$I_\nu(z,\mu) = S_\nu - \frac{\mu}{\rho\kappa_\nu}\frac{\partial I_\nu(z,\mu)}{\partial z}$$

$(\rho\kappa_\nu)^{-1}$ is the mean free path. So derivative term is the change in intensity per mean free path. This is small (equivalently, $I_\nu \simeq S_\nu$). An approximation to the intensity is,

$$I_\nu^{(0)}(z,\mu) \approx S_\nu(T) \approx B_\nu(T)$$

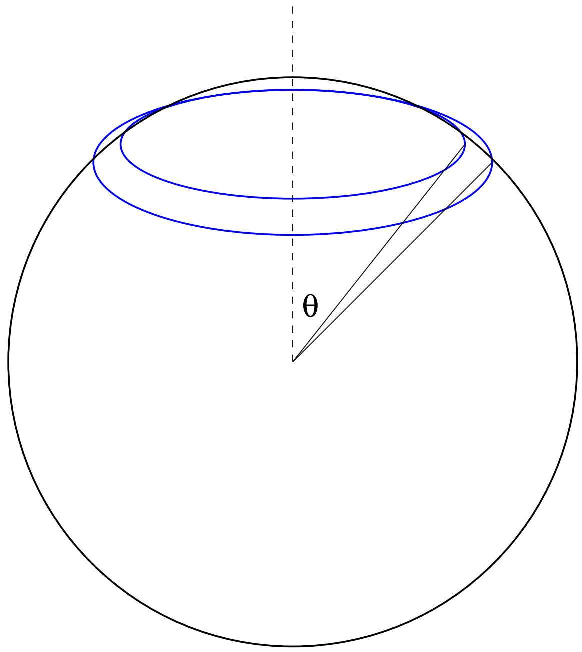
Strategy is to substitute this back to obtain an improved approximation.

Next level of approximation,

$$I_\nu^{(1)}(z,\mu) \approx B_\nu(T) - \frac{\mu}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}$$

Integrate for the flux,

$$F_\nu(z) = \int I_\nu^{(1)}(z,\mu)\cos\theta d\Omega.$$



Solid angle of annulus $d\Omega = 2\pi\sin\theta d\theta = -2\pi d\mu$. So,

$$F_\nu(z) = 2\pi\int_{-1}^{+1} I_\nu^{(1)}(z,\mu)\mu d\mu$$

(minus sign absorbed into limits).

B_ν is isotropic, so only the derivative term in the expression for $I_\nu^{(1)}$ has a non-zero integral.

For the monochromatic flux the integral gives,

$$\begin{aligned} F_\nu(z) &= -\frac{2\pi}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial z}\int_{-1}^{+1}\mu^2 d\mu \\ &= -\frac{4\pi}{3\rho\kappa_\nu}\frac{\partial B_\nu(T)}{\partial z} \\ &= -\frac{4\pi}{3\rho\kappa_\nu}\frac{\partial B_\nu(T)}{\partial T}\frac{\partial T}{\partial z} \end{aligned}$$

which is what we want – relation between the energy flux at frequency ν and the temperature gradient. Remaining work is to put this in a useful form.

The total flux is then,

$$\begin{aligned} F(z) &= \int_0^\infty F_\nu(z) d\nu \\ &= -\frac{4\pi}{3}\frac{\partial T}{\partial z}\int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu(T)}{\partial T} d\nu \end{aligned}$$

Using $F = \pi B$, we can write,

$$\begin{aligned} \int_0^\infty \frac{\partial B_\nu(T)}{\partial T} d\nu &= \frac{\partial}{\partial T}\int_0^\infty B_\nu(T) d\nu \\ &= \frac{\partial B(T)}{\partial T} \\ &= \frac{4\sigma T^3}{\pi} \end{aligned}$$

Still have $(\rho\kappa_\nu)^{-1}$ to deal with. Define the **Rosseland mean opacity**, κ_R, to be,

$$\frac{1}{\rho\kappa_R} \equiv \frac{\int_0^\infty \frac{1}{\rho\kappa_\nu}\frac{\partial B_\nu}{\partial T}d\nu}{\int_0^\infty \frac{\partial B_\nu}{\partial T}d\nu}.$$

With this definition, the energy flux is,

$$F(z) = -\frac{16\sigma T^3}{3\rho\kappa_R}\frac{\partial T}{\partial z}.$$

Properties of the equation of radiative diffusion,

- Flux behaves as an effective heat conductivity.
- Material properties enter via the Rosseland mean – this must be computed for the gas of given metallicity, temperature etc.
- Applies provided that quantities change slowly on the scale of the mean free path (ie, not just plane parallel).

In spherical symmetry,

$$L_r = -\frac{16\pi acr^2}{3\kappa_R\rho}T^3\frac{dT}{dr}.$$

The Eddington limit

The absorption of photons by the medium must lead to a force, as the photons carry momentum. For a spherically symmetric source with luminosity L,

- Energy flux at distance r is $L/(4\pi r^2)$
- Momentum flux is $L/(4\pi c r^2)$.

The force per unit mass, f_{rad} is then,

$$f_{rad} = \frac{\kappa L}{4\pi c r^2}$$

where κ is the fraction of the momentum flux absorbed by unit mass. For a point mass M, the inward force due to gravity is,

$$f_{grav} = \frac{GM}{r^2}$$

Radiation pressure balances gravity when $f_{rad} = f_{grav}$,

$$L = \frac{4\pi c GM}{\kappa}$$

At greater luminosities the pressure of radiation exceeds the gravitational force, and gas will be blown away.

Assume opacity is due to **Thomson scattering** by free electrons. Mass scattering coefficient per hydrogen atom is σ_T/m_H, where m_H is the mass of a hydrogen atom and σ_T the Thomson cross-section. This defines the Eddington limit,

$$
\begin{aligned}
L_{Edd} &= \frac{4\pi c G M m_H}{\sigma_T} \\
&= 1.25 \times 10^{38} \left(\frac{M}{M_\odot}\right) \mathrm{erg\ s^{-1}} \\
&= 3.2 \times 10^4 \left(\frac{M}{M_\odot}\right) L_\odot .
\end{aligned}
$$

Assumptions:

- Thomson scattering cross section only. In an ionized gas containing metals or dust other processes will often *add* to κ, reducing the maximum luminosity.
- Spherical symmetry.

Application to massive stars

From *Hansen & Kawaler*, Table 2.1, the luminosity of zero age main sequence stars at high masses is approximately,

$$\left(\frac{L}{L_\odot}\right) \simeq 1.2 \times 10^5 \left(\frac{M}{30\ M_\odot}\right)^{2.4}$$

Eliminating L we find that the Eddington limit is reached for $M \sim 100 M_\odot$. This implies that the formation of very massive stars cannot be spherically symmetric, as continuum radiation pressure would blow away the infalling gas.