Polytropes

The temperature does not appear explicitly in the two physical structure equations:

$$\begin{aligned}
\frac{dP}{dr} &= -\rho\frac{Gm}{r^2} \\
\frac{dm}{dr} &= 4\pi r^2 \rho,
\end{aligned}$$

...only implicitly because P often depends upon T. In some cases though (e.g. the non-relativistic degenerate gas), P is only a function of ρ, and substitution of the $P(\rho)$ relation into the above equations is sufficient to define a solution.

Assume a *polytropic relation* holds throughout the star,

$$P = K\rho^{1+\frac{1}{n}}$$

where K is a constant and n is the *polytropic index.*

Polytropes are useful in two situations:

(1) **The equation of state is really polytropic**. A completely degenerate gas has a polytropic EOS in both the relativistic and non-relativistic limits:

- Non-relativistic: $P \propto \rho^{5/3} \rightarrow n = 3/2$
- Relativistic: $P \propto \rho^{4/3} \rightarrow n = 3$

In these cases K is fixed (some combination of fundamental constants).

(2) **The equation of state plus some additional constraint yields a polytropic relation**. e.g.:

- An isothermal ideal gas with $T = T_0$.

  $$P = \frac{R}{\mu} T_0 \rho$$

  which is a polytrope with $n = \infty$ and $K = RT_0/\mu$.

- A fully convective star. Convection maintains the temperature gradient very close to $\nabla_{ad} = 2/5$ (for an ionized ideal gas, no radiation pressure). Thus $T \propto P^{2/5}$. For μ a constant, $T \propto P/\rho$ $\rightarrow P \propto \rho^{5/3}$. A polytrope with $n = 3/2$.

In these cases, K can vary from star to star.

The Lane-Emden equation

Combine the two structure equations:

$$\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = \frac{d}{dr}(-Gm) = -G4\pi r^2 \rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Substitute: $\rho = \lambda\theta^n$, where λ is a constant, and $\theta = \theta(r)$. Hence,

$$P = K\rho^{1+\frac{1}{n}} = K\lambda^{1+\frac{1}{n}}\theta^{n+1}$$

and we obtain,

$$\left[\frac{(n+1)K\lambda^{\frac{1}{n}-1}}{4\pi G}\right]\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

Finally, rescale r in terms of a new dimensionless variable ξ. Write $r = r_n \xi$, where the scale length r_n is:

$$r_n = \left[\frac{(n+1) K \lambda^{\frac{1}{n}-1}}{4\pi G} \right]^{1/2}$$

Obtain:

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left(\xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n$$

which is the Lane-Emden equation.

Boundary conditions:

- Let $\lambda = \rho_c$. Then at $r = 0$, requirements that $\rho = \rho_c$ and $dP/dr = 0$ yield:

$$\theta(\xi = 0) = 1, \quad \frac{d\theta}{d\xi}(\xi = 0) = 0$$

- Let $\xi = \xi_1$ be the location of the *first zero* of θ. Requirement that $P = \rho = 0$ there yields:

$$\theta(\xi_1) = 0 \quad \text{at}\, \xi = \xi_1$$

Note: the Lane-Emden equation can be numerically integrated by starting from the center and integrating outward until $\theta = 0$, thereby fixing the initially undefined radius.

There are three analytic solutions corresponding to polytropes with finite mass:

- **n = 0**: Solution corresponds to a constant density sphere,

$$\theta_0(\xi) = 1 - \frac{\xi^2}{6}$$

  with $\xi_1 = \sqrt{6}$.

- **n = 1**: Solution is,

$$\theta_1(\xi) = \frac{\sin \xi}{\xi}$$

  with $\xi_1 = \pi$.

- **n = 5**: Solution has a finite central density and mass, but the radius is unbounded,

$$\theta_5(\xi) = \left[1 + \frac{\xi^2}{3}\right]^{-1/2}$$

  with $\xi_1 \to \infty$

I doubt you will ever need to consider polytropes *apart* from the $n = 3/2$ and $n = 3$ cases. So these analytic solutions are of limited utility.