Thermodynamics of a partially ionized gas

Thermodynamics of partially ionized gases differs from fully ionized case because:

- Number of free particles is not constant.
- Ionization energy is required to increase the number of free particles.

Near the ionization point, Saha equation implies that small changes in $T \rightarrow$ large changes in the degree of ionization. Therefore expect that c_V and c_p will be large for a hydrogen gas at $T \simeq 10^4$ K.

Consider partially ionized pure hydrogen.

$$c_V = \left(\frac{dQ}{dT}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V$$

Let N, H, H^+ represent number / unit volume of free particles, neutral H, and ionized H. The internal energy per unit mass is approximately,

$$U(T, V) = \frac{3}{2} NkTV + \chi_H H^+ V$$

with $V = 1/\rho$. Note: this is an approximation, but a good one.

Specific heat capacity at constant volume,

$$c_V = \left(\frac{\partial U}{\partial T}\right)_V = \frac{3}{2}NkV + \frac{3}{2}kTV\left(\frac{\partial N}{\partial T}\right)_V + \chi_H V\left(\frac{\partial H^+}{\partial T}\right)_V$$

Add the usual constraints for pure H. Charge neutrality,

$$N_e = H^+$$

Total number density of free particles,

$$N = H + H^+ + N_e = H + 2H^+$$

Relation between the specific volume and the number densities (neglecting the electron mass),

$$H^+ + H = \frac{N_A}{V}$$

Hence,

$$\left(\frac{\partial N}{\partial T}\right)_V = \left(\frac{\partial H^+}{\partial T}\right)_V = -\left(\frac{\partial H}{\partial T}\right)_V$$

The specific heat is then,

$$c_V = \frac{3}{2} NkV \left[1 - \frac{2T}{3N}\left(\frac{3}{2} + \frac{\chi_H}{kT}\right)\left(\frac{\partial H}{\partial T}\right)_V\right]$$

First term is specific heat capacity for a constant number of particles. Since $(\partial H/\partial T)_V$ is negative, second term represents increase due to ionization.

By writing the Saha equation in the form,

$$\frac{H^+ N_e}{H} = \frac{(N_A/V - H)^2}{H} = \left(\frac{2\pi m_e kT}{h^2}\right)^{3/2} e^{-\chi_H/kT}$$

can calculate the partial derivative. In terms of the number densities of H and H^+ find,

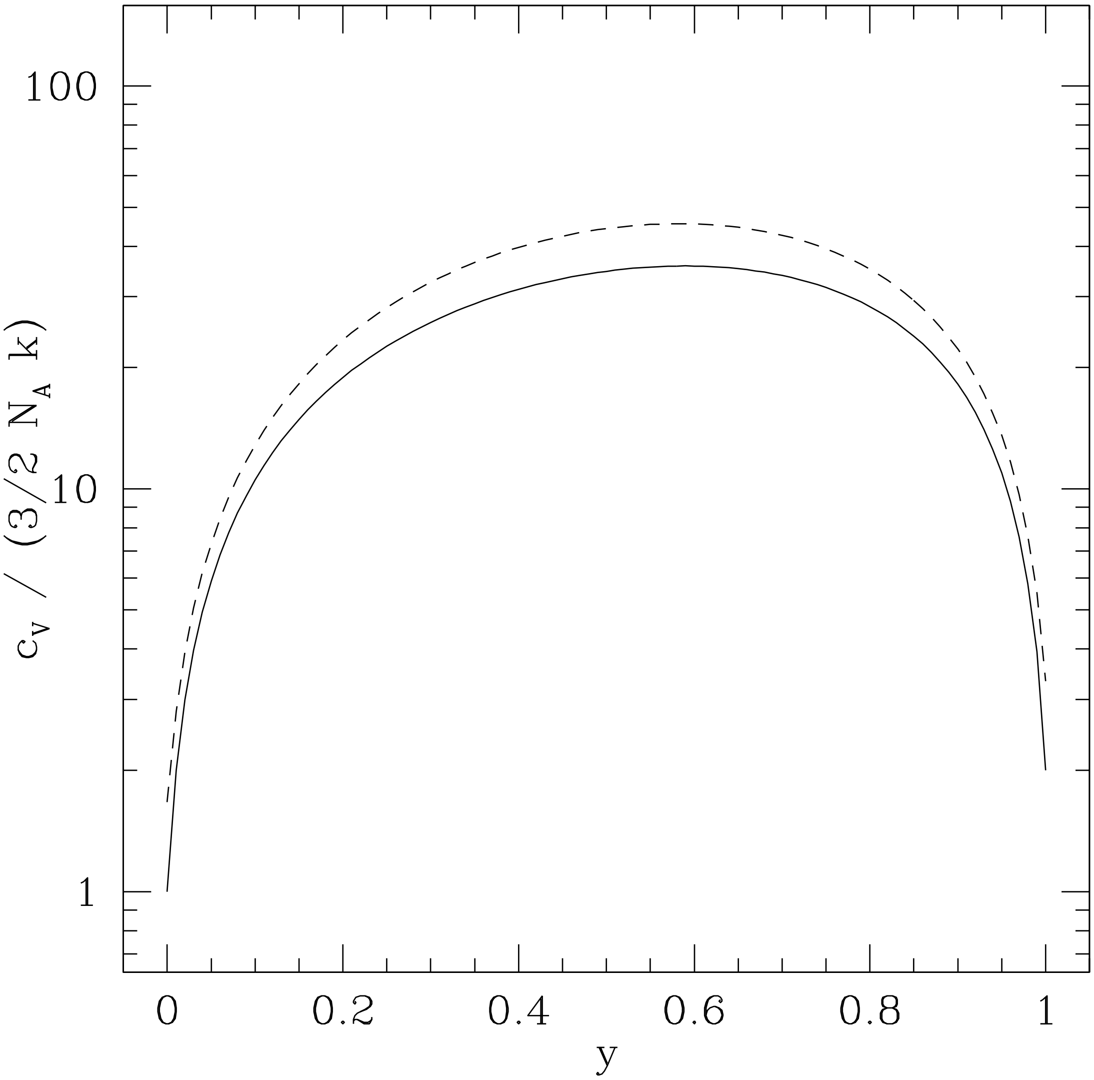
$$c_V = c_V^{(0)} \left[1 + \frac{2}{3}\left(\frac{3}{2} + \frac{\chi_H}{kT}\right)^2 \frac{H^+ H}{(H + 2H^+)(2H + H^+)}\right]$$

and similarly,

$$c_p = c_V^{(0)} \left[\frac{5}{3} + \frac{1}{3}\left(\frac{5}{2} + \frac{\chi_H}{kT}\right)^2 \frac{H^+ H}{(H + H^+)^2}\right]$$

where $c_V^{(0)} = 3NkV/2$. Note: brackets are symmetric in degree of ionization, but $c_V^{(0)}$ is not.

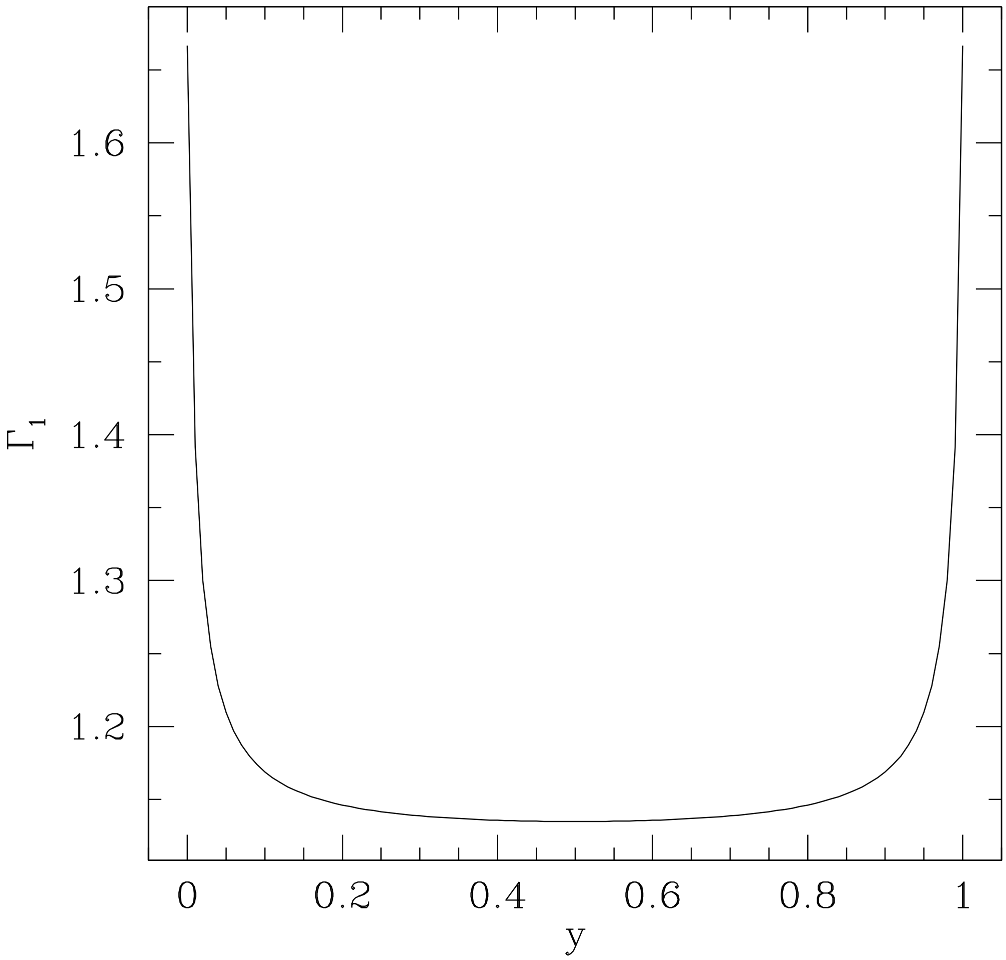
Evaluating these expressions (approximately) assuming that $T = 10^4$ K throughout ionization. As before y is the ratio of ionized gas to total number density.



- Solid line: c_V. Dashed line: c_P.
- Peak values are around 30 times the value for neutral hydrogen.

Adiabatic exponents

Clayton (p. 125) derives the adiabatic exponent Γ_1 as a function of the degree of ionization. Derivation is very similar to what we've done already, and gives result for low density:



All three Γ's can be significantly less than 4/3 for hydrogen that is partially ionized (see *Hansen & Kawaler* p. 143 for effect of density).

Physical relevance

Reason for lowered Γ is that ionization is a strong function of temperature. Consider Γ_3,

$$\Gamma_3 - 1 = \left(\frac{\partial \ln T}{\partial \ln \rho}\right)_{ad}.$$

If the gas is compressed, it will tend to heat up. This leads to a rapid increase in the degree of ionization, for which the only energy source is the thermal energy of the gas. Thus T does not rise as rapidly with ρ as for a neutral gas, and Γ_3 is reduced.

For ordinary stars, presence of ionization zones provides the main potential source of pulsational instability. In outline,

- Damping means that not all stars pulsate.
- Second ionization of He provides most significant driving.
- Require that ionization zones occur in radiative regions.

These requirements point to cool ($T_e < 6300$ K) but luminous stars in an *instability strip* of a few hundred K in width. Most importantly:

- Cepheids: $5 - 10\ M_\odot$ stars burning helium.
- RR Lyrae stars: lower mass, also helium burning