Chemical potential

If concentrations of different species in a gas can change, need to apply constraint of chemical equilibrium. Chemical potential of species i is,

$$\mu_i = \left(\frac{\partial E}{\partial N_i}\right)_{S,V}.$$

Consider changing concentrations by dN_i via some reaction. Thermodynamic equilibrium requires,

$$\sum_i \mu_i dN_i = 0$$

We can write any reaction in the form,

$$\sum_i \nu_i C_i = 0$$

where the ν_i are the stoichiometric coefficients and the C_i represent the reacting components. e.g. for the reaction,

$$H^+ + e^- \leftrightarrow H^0$$

$C_1 = H^+$ etc, $\nu_1 = \nu_2 = 1$, $\nu_3 = -1$.

If N_1 changes by some amount dN_1, the changes in the other N_i depend upon the ν_i,

$$\frac{dN_i}{\nu_i} = \frac{dN_1}{\nu_1}$$

Thus we have,

$$\sum_i \mu_i dN_i = \sum_i \mu_i \frac{dN_1}{\nu_1} \nu_i = 0$$

Since dN_1 is arbitrary,

$$\sum_i \mu_i \nu_i = 0$$

The *equation for chemical equilibrium.*

Neutron stars

Standard reference: 'Black holes, white dwarfs and neutron stars' (Shapiro & Teukolsky).

At low T, degenerate EOS with Coulomb corrections applies up to densities of $\sim 10^7$ g cm^{-3}. First important deviation due to inverse β-decay,

$$e^- + p \rightarrow n + \nu$$

which requires e^- to have enough energy to balance the mass difference between proton and neutron,

$$(m_n - m_p)c^2 = 1.29 \text{ MeV}$$

This reaction will transform protons into neutrons **if** ordinary β-decay,

$$n \rightarrow p + e^- + \bar{\nu}$$

is suppressed. This occurs when ρ is high enough because then there are no free states in the Fermi sea for the electron to occupy.

Aim: calculate the required density.

Consider a gas of free electrons, protons, and neutrons in equilibrium. For inverse β-decay, chemical potentials satisfy,

$$\mu_e + \mu_p = \mu_n$$

Define, as for degenerate electrons, parameters,

$$x_e = \frac{p_F^e}{m_e c}, \quad x_n = \frac{p_F^n}{m_n c}, \quad x_p = \frac{p_F^p}{m_p c}$$

We previously showed for degenerate electrons that,

$$\mu_e = m_e c^2 (1 + x_e^2)^{1/2},$$

so chemical potential formula becomes,

$$m_e (1 + x_e^2)^{1/2} + m_p (1 + x_p^2)^{1/2} = m_n (1 + x_n^2)^{1/2}$$

Charge neutrality $n_e = n_p$ implies,

$$\frac{8\pi}{3} \left(\frac{h}{m_e c} \right)^{-3} x_e^3 = \frac{8\pi}{3} \left(\frac{h}{m_p c} \right)^{-3} x_p^3,$$

i.e. $m_e x_e = m_p x_p$.

To find minimum density for neutrons to appear, set $x_n = 0$. Then

$$m_e(1 + x_e^2)^{1/2} = m_n - m_p(1 + x_p^2)^{1/2}$$

Assume (correctly) that protons are non-relativistic so $x_p \ll 1$. Then,

$$m_e(1 + x_e^2)^{1/2} = m_n - m_p$$

Solving for x_e and writing in terms of n_e,

$$n_e = \frac{8\pi}{3}\left(\frac{h}{m_e c}\right)^{-3}\left[\left(\frac{m_n - m_p}{m_e}\right)^2 - 1\right]^{3/2}$$

In terms of physical density,

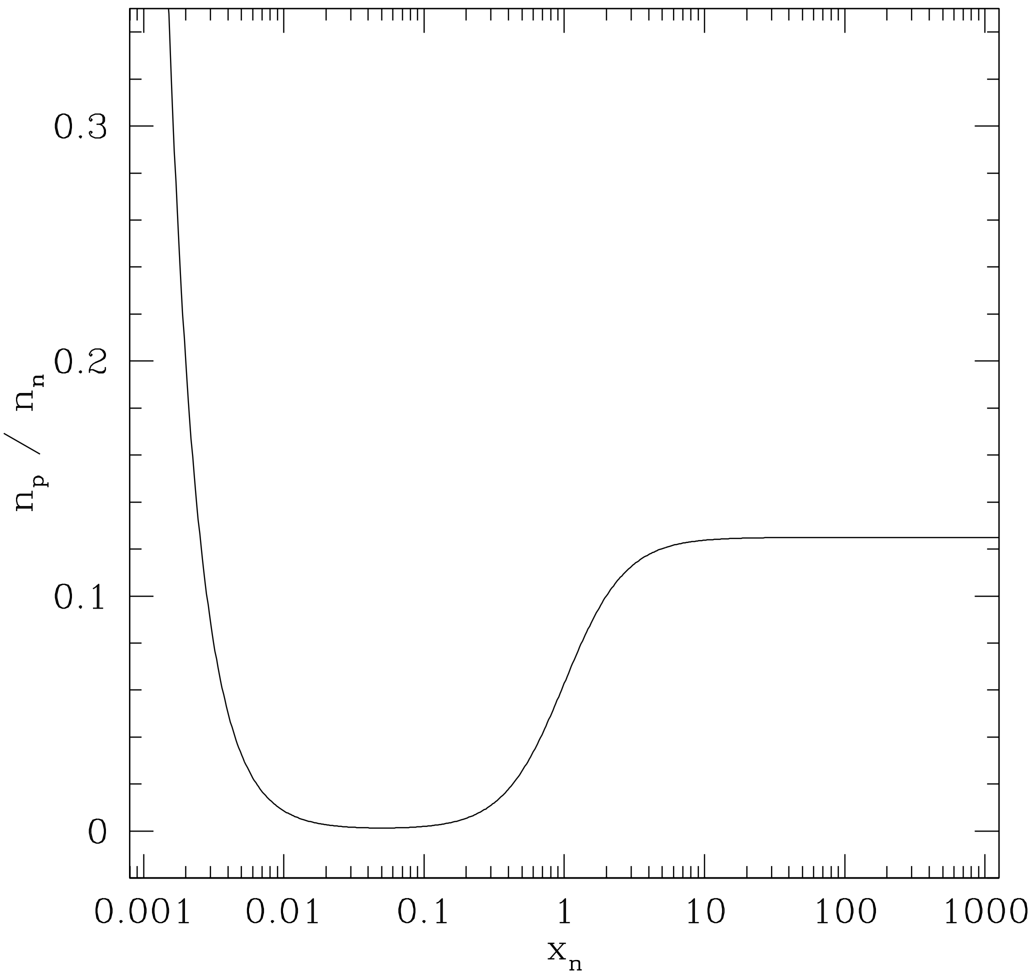
$$\rho \approx n_e m_p \approx 1.2 \times 10^7 \text{ g cm}^{-3}$$

Above this density, equilibrium composition has increasing proportion of neutrons. Substitute expression for charge neutrality into chemical potential formula to find,

$$\frac{n_p}{n_n} = \left(\frac{m_p x_p}{m_n x_n}\right)^3 \simeq \frac{1}{8}\left[\frac{1 + 4Q/m_n x_n^2 + 4(Q^2 - m_e^2)/m_n^2 x_n^4}{1 + 1/x_n^2}\right]^{3/2}$$

where $Q = m_n - m_p$.

Plot this ratio as $f(x_n)$ (i.e. with increasing density):



Minimum value occurs at $\rho \approx 8 \times 10^{11}$ g cm^{-3}. Asymptotic value is 1/8 as $\rho \to \infty$.

Strictly, these results apply to an equilibrium system of fixed charge (zero), baryon number, and lepton number. We have considered a limit where $n_\nu \to 0$.

As with Coulomb effects at lower densities, complications arise because of interactions. At 'low' densities, minimum energy state ('nuclear equilibrium') is bound nuclei. Calculation of the EOS requires knowledge of the binding energy (or $M(Z, A)$) of different nuclei.

Summary:

- Below $\rho \sim 10^7$ g cm^{-3}, equilibrium nuclide is $^{56}_{26}$Fe.
- At higher densities, inverse β-decay alters equilibrium nuclear composition towards more neutron rich matter. Nuclei are stabilized against β-decay by the filled Fermi sea of electrons.
- At $\rho \sim 4 \times 10^{11}$ g cm^{-3} reach **'neutron drip'**. Most stable equilibrium is two-phase: electrons and nuclei plus free neutrons.
- Pressure is dominated by fully relativistic electrons up to densities of order 4×10^{12} g cm^{-3}. Neutrons thereafter.
- Above nuclear densities ($\rho \sim 2.8 \times 10^{14}$ g cm^{-3}), substantial uncertainties in the EOS remain. Relevant to the radii and structure of neutron stars.

Physical idea: **nuclear equilibrium** – energy cannot be lowered by changing composition via strong, weak or EM interactions.

Modern reference: Shen et al., Prog. Theoretical. Phys., **100**, 1013 (1998).

Specific heats and adiabatic exponents

Consider first an ideal nondegenerate gas. Calculation of the specific heats and adiabatic behavior is a textbook exercise. Start with,

$$PV = \frac{R}{\mu}T$$

where V is the volume of unit mass of gas. First law of thermodynamics is,

$$dQ = dU + PdV$$

For a perfect gas, $U = U(T)$, so equivalently,

$$dQ = \frac{dU}{dT}dT + PdV$$

Specific heat at constant volume,

$$c_V \equiv \left(\frac{dQ}{dT}\right)_V = \frac{dU}{dT}$$

To get the specific heat at constant pressure, use the EOS,

$$PdV + VdP = \frac{R}{\mu}dT$$

Using this,

$$dQ = \left(\frac{dU}{dT} + \frac{R}{\mu}\right) dT - V dP$$

so,

$$c_p = c_V + \frac{R}{\mu}$$

The ratio of the specific heats c_p/c_V, depends upon the number of degrees of freedom of the gas. For f degrees of freedom,

$$\gamma = \frac{c_p}{c_V} = 1 + \frac{2}{f}$$

so in stellar interiors where the gas is ionized (no internal degrees of freedom) $f = 3$ and $\gamma = 5/3$.

Consider adiabatic changes. For an ideal gas, the first law can be written,

$$dQ = c_V dT + \frac{RT}{\mu V} dV.$$

An adiabatic change is defined as a quasistatic change of state in which no heat is added. Setting $dQ = 0$ and substituting for the specific heats gives,

$$c_V \frac{dT}{T} + (c_p - c_V)\frac{dV}{V} = 0.$$

For an ideal gas, the specific heats are constants. Integrating,

$$TV^{\gamma-1} = \text{constant}$$

Likewise we obtain,

$$\begin{aligned} TV^{\gamma-1} &= \text{constant} \\ PV^{\gamma} &= \text{constant} \\ P^{1-\gamma}T^{\gamma} &= \text{constant} \end{aligned}$$

In differential form, adiabatic changes satisfy,

$$\begin{aligned} \frac{dT}{T} + (\gamma - 1)\frac{dV}{V} &= 0 \\ \frac{dP}{P} + \frac{\gamma}{1-\gamma}\frac{dT}{T} &= 0 \\ \frac{dP}{P} + \gamma\frac{dV}{V} &= 0 \end{aligned}$$

These relations hold for ideal gases. For the important case of gas + radiation, a **different** 'γ' is required for each of these relations. These γ's are not constants, nor do they equal the ratio of specific heats.