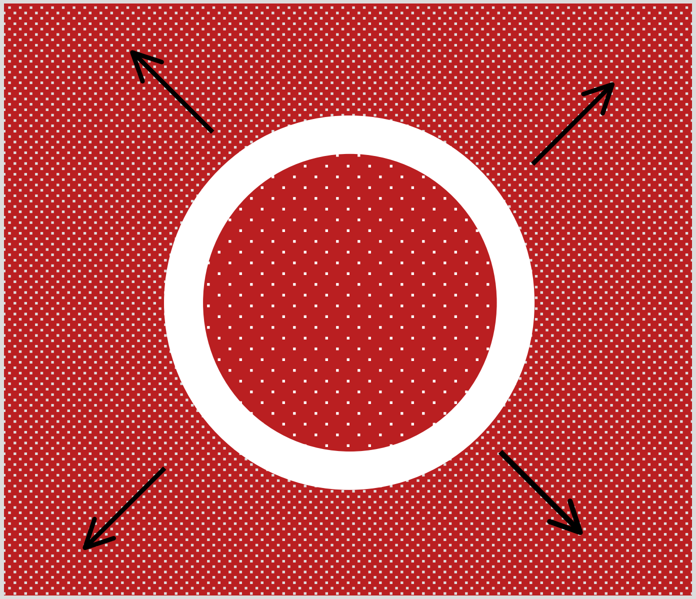
Structure formation

Consider a **flat**, **matter dominated universe**
Ignore the cosmological constant - it wasn't (very) important at high z when the first structures were forming

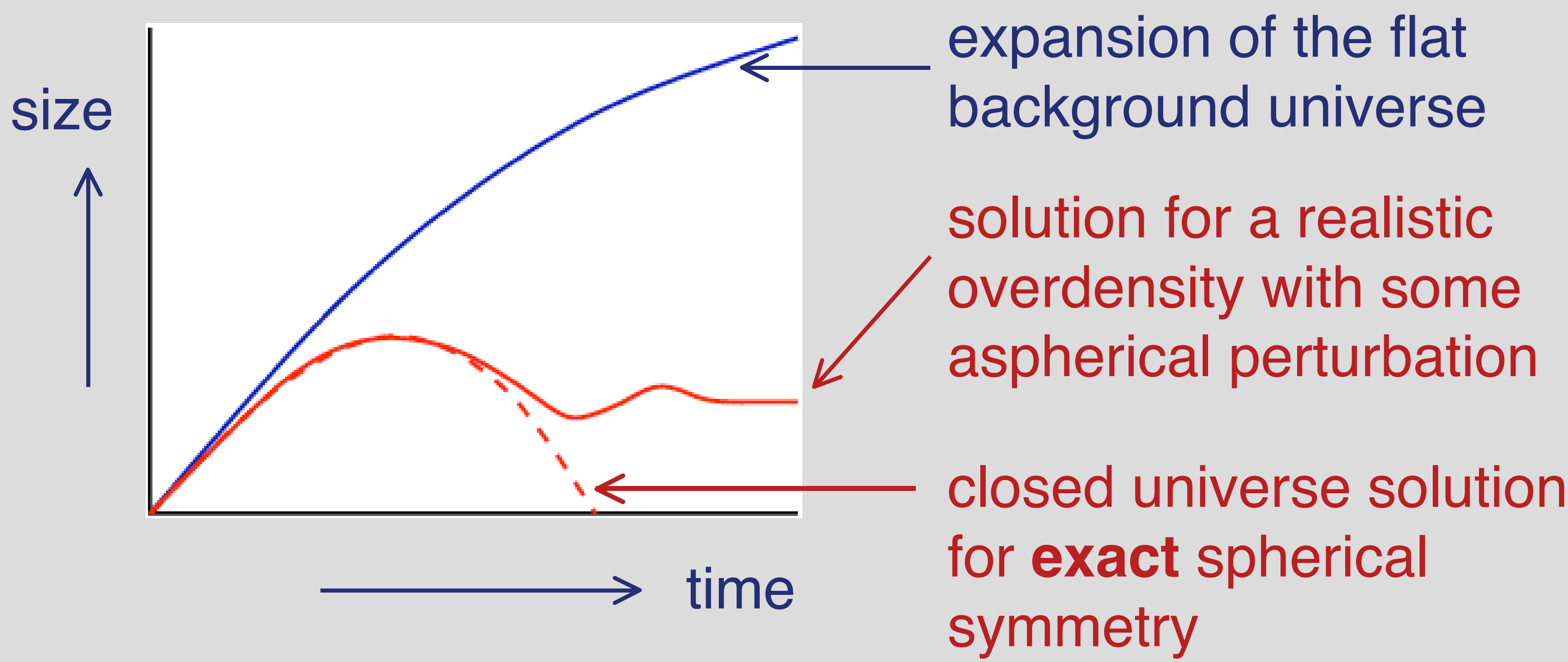


Imagine a spherical volume of the universe which is slightly denser than the background.

How will this overdense region evolve with time as the Universe expands?

Recall **Birkhoff's theorem** - gravitational force inside a sphere depends **only** on the matter inside

⇒ **Overdense region behaves exactly like a small closed Universe!**



Schematic evolution:

- Density contrast grows as universe expands
- Perturbation `turns around' at $R = R_{turn}$, $t = t_{turn}$
- If exactly spherical, collapses to a point at $t = 2\ t_{turn}$
- Realistically, bounces and **virializes** at radius $R = R_{virial}$

Can use the virial theorem (textbook Section 3.1) to derive the final radius of the collapsed perturbation.

Let perturbation have mass M, kinetic energy <KE>, and gravitational potential energy <PE>.

Virial theorem: <PE> + 2 <KE> = 0 (in equilibrium)
Energy conservation: <PE> + <KE> = constant

$$-\frac{GM^2}{R_{turn}} = -\frac{GM^2}{R_{virial}} + \langle KE \rangle = -\frac{GM^2}{2R_{virial}}$$

energy at turnaround → $-\frac{GM^2}{R_{turn}}$

energy when virialized → $-\frac{GM^2}{R_{virial}} + \langle KE \rangle$

applying virial theorem → $-\frac{GM^2}{2R_{virial}}$

Note: ignore prefactor in potential energy here

Conclude: $R_{virial} = \frac{1}{2} R_{turn}$

How dense are collapsed objects?

Can apply the solution for a closed universe to calculate the final overdensity in a spherical collapse model:

Friedmann equation:

$$\dot{a}^2 = \frac{A^2}{a} - kc^2$$ where constant A: $$A^2 = \left(\frac{8\pi G}{3}\right)\rho_0 a_0^3$$

Solutions:

$$a = \left(\frac{3A}{2}\right)^{2/3} t^{2/3}$$

k = 0 solution derived previously

$$a = \frac{1}{2}\frac{A^2}{c^2}(1 - \cos\theta)$$

$$t = \frac{1}{2}\frac{A^2}{c^3}(\theta - \sin\theta)$$

Parametric solution for a closed, k = 1 universe (not derived here)

Calculate the turnaround time for the collapsing sphere by finding when the size of the small closed universe has a maximum:

$$\frac{da}{d\theta} = \frac{1}{2}\frac{A^2}{c^2}\sin\theta = 0 \quad \Rightarrow \quad \theta = 0, \pi, \ldots$$

At turnaround, $\theta = \pi$, which corresponds to time,

$$t_{turn} = \frac{1}{2}\frac{A^2}{c^3}\pi$$

The scale factor of the perturbation and of the background universe are just:

$$a_{sphere} = \frac{1}{2}\frac{A^2}{c^2}(1 - \cos\pi) = \frac{A^2}{c^2}$$

$$a_{background} = \left(\frac{3A}{2}\right)^{2/3} t_{turn}^{2/3} = \left(\frac{3\pi}{4}\right)^{2/3}\frac{A^2}{c^2}$$

The density contrast at turnaround is therefore:

$$\frac{\rho_{sphere}}{\rho_{background}} = \left(\frac{a_{sphere}}{a_{background}} \right)^{-3} = \frac{9\pi^2}{16}$$

At the time when the collapsing sphere virialized, at $t = 2\ t_{turn}$:

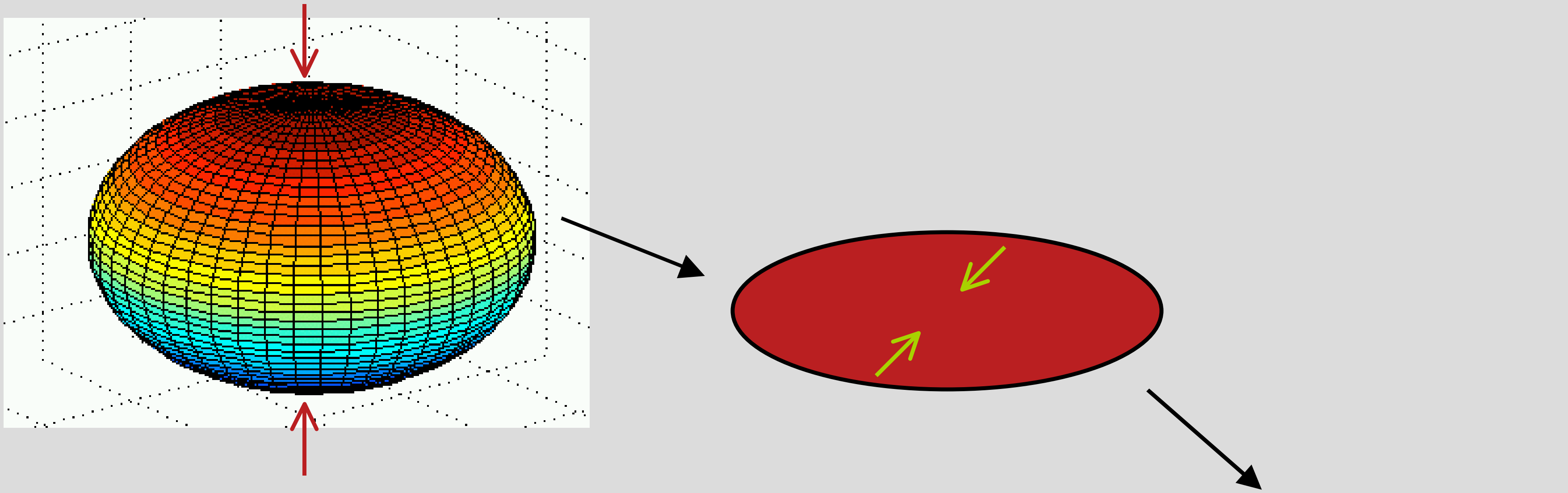
- Its density has increased by a factor of 8
- Background universe's density has decreased by a factor of $(2^{2/3})^3 = 4$

Final result: a collapsing object virializes when its density is greater than the mean density of the universe by a factor of $18\pi^2 \sim 180$…

First objects to form are small and dense - these later merge to form larger structures - `bottom-up' structure formation.

Non-spherical collapse

Real perturbations will not be spherical - can gain qualitative insight into real behavior by considering collapse of an ellipsoidal overdensity:



Fastest collapse in the shortest direction:

- Perturbation first pancakes
- Then forms **filaments**

Filamentary structure is seen both in numerical simulations of structure formation and in galaxy redshift surveys.

Cone diagram: 3-degree slice