Summary: expansion of the Universe

Derived an equation for the acceleration of the Universe:

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho_m + \frac{3P}{c^2}\right)a + \frac{\Lambda}{3}a$$

acceleration of the **scale factor**

density

pressure

cosmological constant

Important points: **pressure has a gravity**, in the absence of a cosmological constant the expansion decelerates

Need to:

- Understand how different terms scale with a
- Integrate twice
- Interpret the constant of integration in terms of the **curvature** of the Universe

Simplest case: Matter dominated Universe

If radiation is negligible, and there is no cosmological constant, then the acceleration equation is just:

$$\ddot{a} = -\frac{4\pi G}{3}\rho_m a$$

Density changes as the Universe expands. If at some time when $a = a_0$ the density is $\rho_m = \rho_0$, then:

$$\rho_m = \rho_0 \frac{a_0^3}{a^3}$$

Substituting:

$$\ddot{a} = -\underbrace{\frac{4\pi G \rho_0 a_0^3}{3}}_{\text{constants}} \frac{1}{a^2}$$

Integrating this expression once with respect to time:

$$\dot{a}^2 = \frac{8\pi G\rho_0 a_0^3}{3}\frac{1}{a} - kc^2.$$

↑ constant of integration

How do we interpret k? For a Universe that is expanding, note that:

- RHS must be greater than zero. If $k > 0$, then there is a maximum value of a at which the expansion turns around and collapse begins. Described as a **closed Universe**.
- If $k < 0$, expansion rate eventually tends to a constant determined by value of k - an **open Universe**.
- If $k = 0$, expansion slows but never reverses - a **flat Universe**.

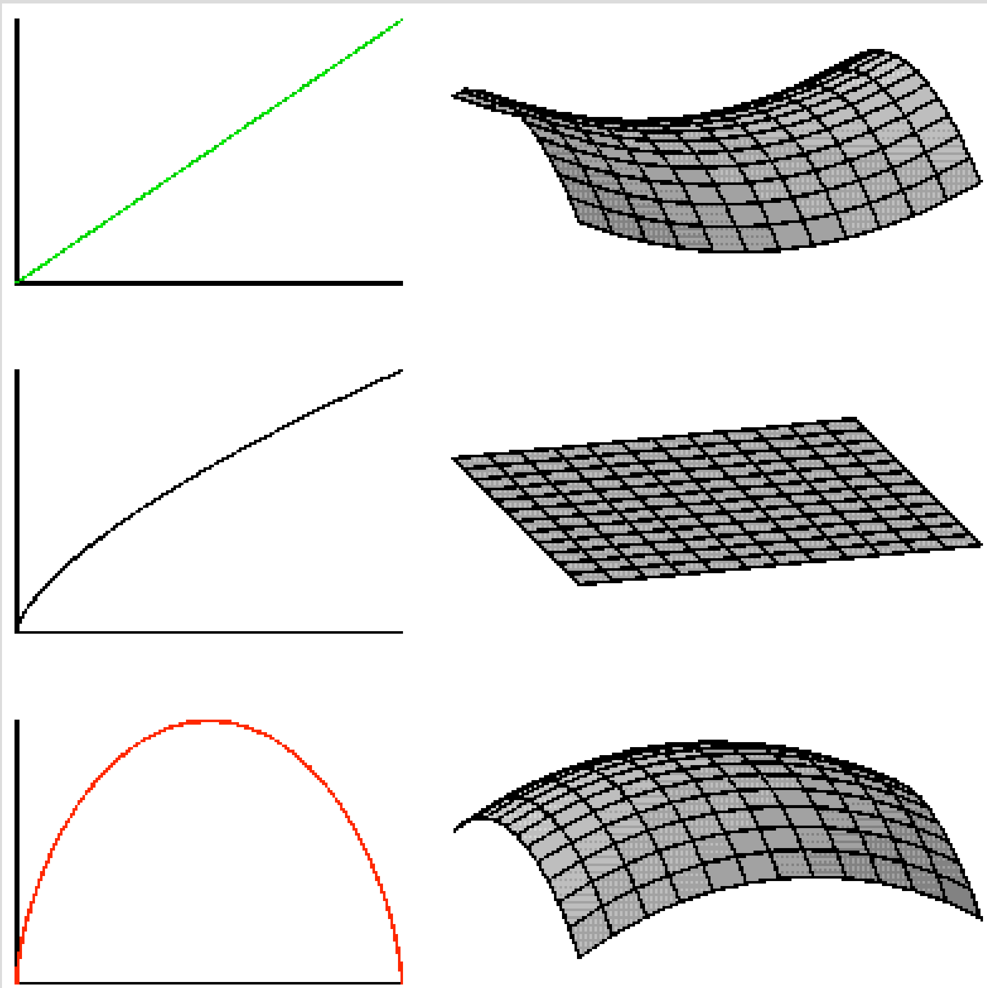
More compete treatment (see textbook, or, better, *Relativity and Cosmology* course), identifies k with the geometry of the Universe. Spatial part of the metric is:

$$ds^2 = a(t)^2 \left[\frac{dR^2}{1 - kR^2} + R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2 \right]$$

(note: conventional to redefine coordinates so k takes values of k = 0, k = +1, k = -1 only)

If k = 0, this is just the usual Euclidean expression for the distance between two points, written in polar coordinates.

For open or closed Universes, the geometry is not so simple. Fortunately, WMAP results strongly suggest that the Universe is flat (or very close to it…)



Open Universe

Flat Universe

Closed Universe

Critical density

Apply the matter dominated solution to the current epoch. Put a = a_0:

$$\dot{a}^2 = \frac{8\pi G \rho_0 a_0^3}{3} \frac{1}{a} - kc^2.$$

$$\frac{kc^2}{a_0^2} = \frac{8\pi G}{3} \rho_0 - \left(\frac{\dot{a}_0}{a_0} \right)^2$$

This is just the Hubble constant squared

Result: a flat, matter-dominated Universe requires that the density equal the **critical density**, given by,

$$\rho_c \equiv \frac{3H_0^2}{8\pi G}$$

For a Hubble constant of 70 km s^{-1} Mpc^{-1}, obtain:

$$\rho_c \approx 9 \times 10^{-30} \text{ g cm}^{-3}$$

Extremely common to measure densities as fractions of the critical density, via a parameter Ω:

$$\Omega \equiv \frac{\rho}{\rho_c}$$

← ρ can be density of Universe, or just some component - e.g. baryons, stars, microwave background radiation etc…

For a flat Universe, easy to obtain the evolution of the scale factor with time:

$$\frac{da}{dt} = \frac{A}{a^{1/2}}$$

absorb the constants into A

$$\int a^{1/2} da = A \int dt$$

$$\frac{2}{3} a^{3/2} = At$$

constant of integration is zero since (by definition of t) can choose a = 0 at t = 0

$$a = \left(\frac{3A}{2} \right)^{2/3} t^{2/3}$$

Age of the Universe

For this model, the age of the Universe is:

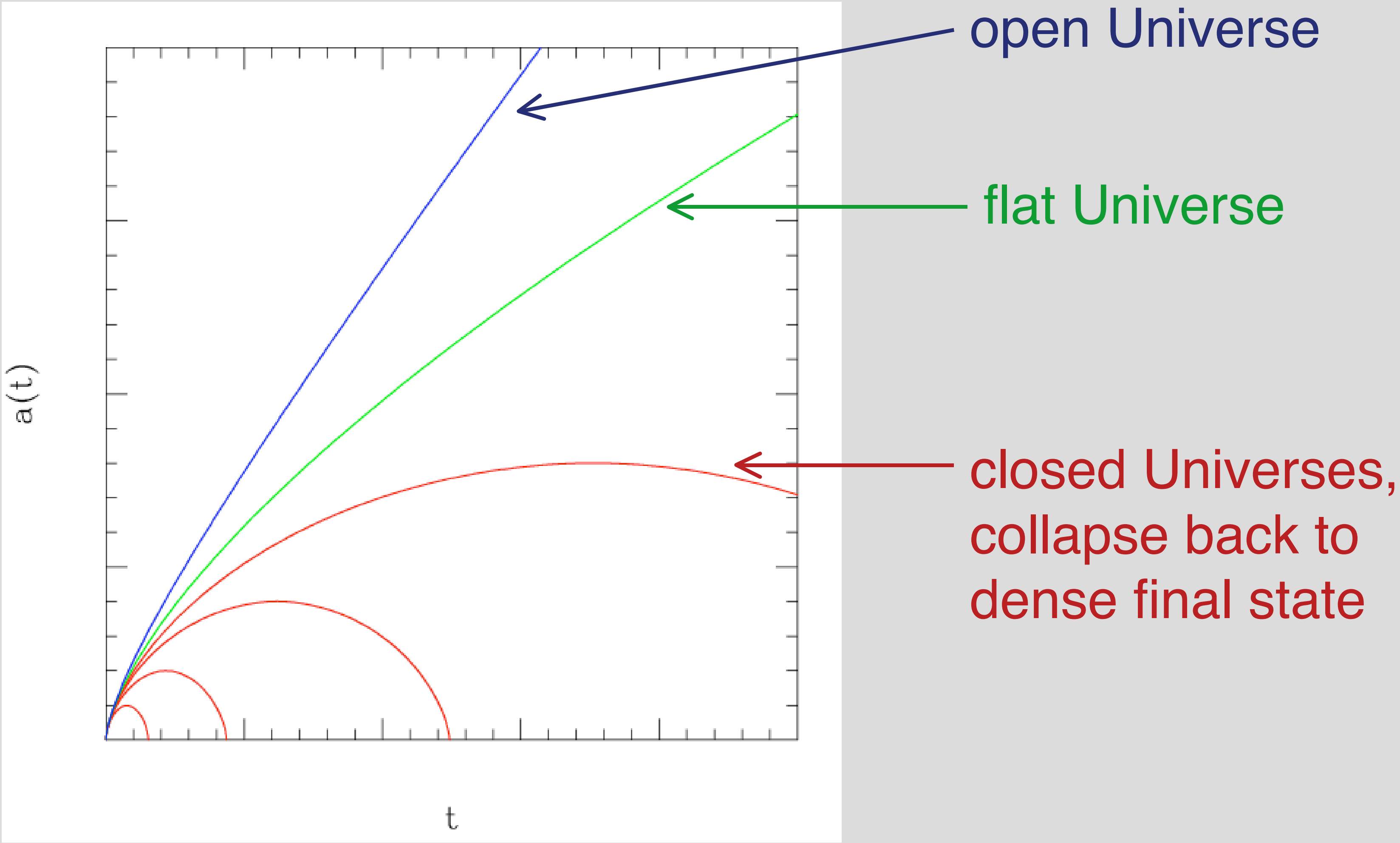
$$t = \frac{2}{3}\frac{a^{3/2}}{A} = \frac{2}{3}\frac{a^{3/2}}{\dot{a}a^{1/2}} = \frac{2}{3}H_0^{-1}$$

Taking a Hubble constant of 70 km s^{-1} Mpc^{-1} again, the age of the Universe would be 9.3 Gyr.

This is too small - the ages of stars appear to be greater than 10 Gyr in some instances…

a(t) for open, flat and closed Universes

With more effort, can obtain a(t) for open and closed Universes as well. Result:



Universes with non-zero cosmological constant

If we restore the cosmological constant to the acceleration equation, but continue to ignore radiation, then integrating once we obtain for k = 0:

$$\dot{a}^2 = \frac{A^2}{a} + \frac{1}{3}\Lambda a^2$$

This is **Friedmann's equation** for the special case of a **flat** Universe containing **matter** plus a **cosmological constant**.

Note: when the Universe is young and small, expect the first term to dominate - at early times expect the evolution will be essentially identical to previous model.

Simplest case of all is an `empty' Universe - in which the matter contributes negligibly to the expansion. In this case A = 0. Integrating:

$$\frac{da}{dt} = \sqrt{\frac{\Lambda}{3}}a$$

$$\frac{da}{a} = \sqrt{\frac{\Lambda}{3}}dt$$

$$a = a(t = 0)e^{\sqrt{\frac{\Lambda}{3}}t}$$

Positive cosmological constant drives **exponential** rate of expansion of the Universe.

Any small amount of matter present rapidly becomes enormously diluted by the expansion.

Considerable range of behavior is possible in Universes with a cosmological constant. Most useful solution is one describing a Universe which is:

- Flat
- Has a positive cosmological constant
- Began in a big bang - i.e. a = 0 at t = 0

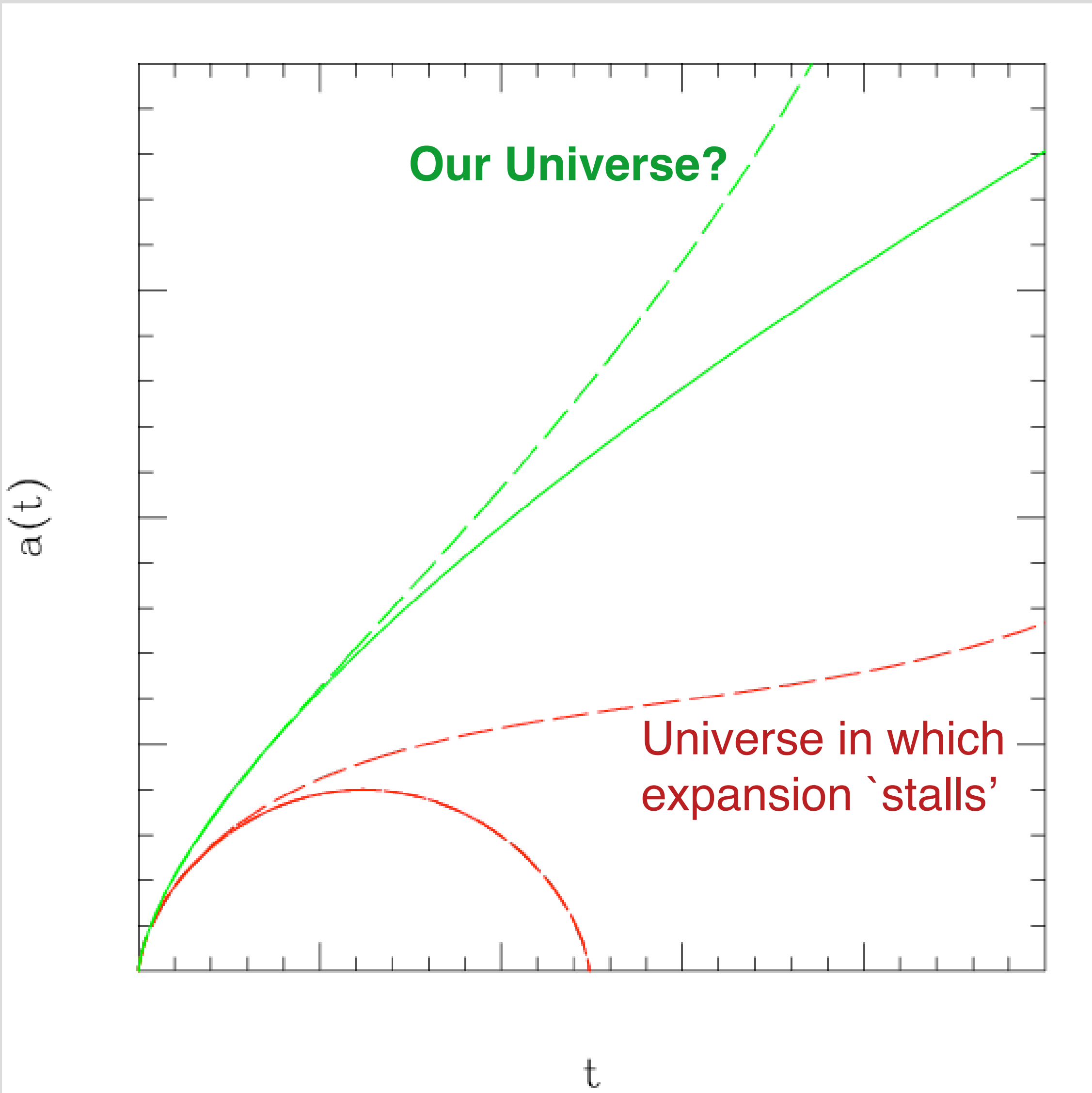
In this case, can show that:

$$a = \left(\frac{3A^2}{2\Lambda} \right)^{1/3} \left[\cosh\left(\sqrt{3\Lambda}\, t \right) - 1 \right]^{1/3}$$

Universe:

- Expands forever
- Expands as $t^{2/3}$ at early times

Behavior of a(t) with a cosmological constant



General case is now complicated. Flat universe is **not** now equivalent to one in which the expansion slows to a halt…

Solid lines: no □
Dashed lines: □ > 0