Expansion of the Universe

Homogeneity + isotropy ⇒ Expansion of Universe cannot alter the relative orientations of galaxies expanding with the Universe

Means that if the present separation between two galaxies is d_0, then the separation at time t can be written as:

$$d = d_0 a(t)$$

a(t) is the **scale factor** - it is dimensionless and depends upon time but **not** on position.

Relative velocity of the two galaxies is:

$$v = \dot{d} = d_0 \dot{a}(t) = \frac{\dot{a}}{a} d$$

Definition of the Hubble parameter is v = H x d, so:

$$H = \frac{\dot{a}}{a}$$

H is a function of time, present value is denoted H_0

Sometime useful to define **comoving coordinates**. If we divide distances by a(t), then two galaxies which simply recede from each other due to the Hubble expansion always have the same separation in comoving coordinates.

Can derive the evolution of a(t) using mostly Newtonian mechanics, provided we accept two results from General Relativity:

1) **Birkhoff's theorem**: this states (in part) that for a spherically symmetric system, the force due to gravity at radius r is determined only by the mass *interior* to that radius.

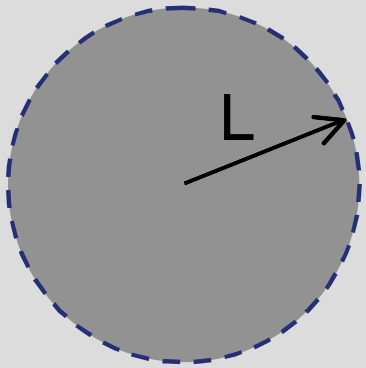
2) **Energy** contributes to the gravitating mass density, which equals:

$$\rho_m + \frac{u}{c^2}$$

ρ_m: density of matter

u: energy density (ergs cm^{-3}) of radiation and relativistic particles

Consider the evolution of a spherical volume of the Universe, radius L:



Sphere expands with the Universe, so $L = L_0 a(t)$

Since expansion is described entirely by a(t), can consider any size sphere we want - if L is small `reasonable' to assume that space is approximately Euclidean.

Expansion of the sphere will slow due to the gravitational force of the matter (+energy) inside:

$$\frac{d^2L}{dt^2} = -\frac{GM}{L^2}$$

Note: no pressure forces because Universe is homogenous

Contributions to the gravitating mass come from matter plus energy density from radiation:

- Matter density ρ_m
- Radiation with energy density u has pressure:

$$P = \frac{1}{3} u$$

gravitating mass density is:

$$\rho = \rho_m + \frac{3P}{c^2}$$

Mass within sphere, radius L, is:

$$M = \rho V = \frac{4}{3} \pi L^3 \rho$$

Substitute into acceleration equation:

$$\frac{d^2L}{dt^2} = -\frac{G}{L^2}\left(\frac{4}{3}\pi L^3 \rho\right)$$

Since L = $L_0 a(t)$, with L_0 a constant, can write this as an equation for the evolution of the scale factor a(t):

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho_m + \frac{3P}{c^2}\right)a$$

(also substituting for ρ in the above expression)

- Matter density $\rho_m > 0$
- Pressure of radiation is also positive

RHS of the equation is always **negative**
Impossible to have a **static Universe**

Lack of static solutions is not a problem - Universe is expanding. But this was not known in 1917. Einstein therefore modified the equations of General Relativity so the equation becomes:

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho_m + \frac{3P}{c^2}\right)a + \frac{\Lambda}{3}a$$

Λ is the **cosmological constant** (the factor 3 is just convention). A positive cosmological constant tends to accelerate the expansion - i.e. as if the Universe is filled with material with a negative pressure.

Is a static solution stable?

Properties of the cosmological constant

Cosmological constant is assumed to be a *smooth* component… i.e. it does not cluster or clump together in the same way as ordinary matter.

Original cosmological constant was… **constant** in time! This is just an assumption, however - models in which the vacuum energy varies with time are called **quintessence**.

For Λ to be important today, it must have a value comparable to the first term in the equation:

$$\Lambda \sim 4\pi G\rho_m \sim 10^{-36}\,\mathrm{s}^{-2} \qquad \text{(for } \rho \sim 10^{-30}\text{ g cm}^{-3}\text{)}$$

`Fundamental' unit of time is the Planck time: $t_{Planck} = \sqrt{\frac{Gh}{c^5}} = 10^{-43}\ \mathrm{s}$

Might guess that $\Lambda \sim t_{Planck}^{-2}$ …bad guess by factor 10^{120}.

Cosmological constant problem…

Which terms are most important?

$$\ddot{a} = -\frac{4\pi G}{3}\left(\rho_m + \frac{3P}{c^2}\right)a + \frac{\Lambda}{3}a$$

- Early times - energy density of radiation is large compared to the energy density of matter
- Later, matter dominates
- Finally, if Λ is non-zero, eventually it dominates

Radiation dominated

→ Matter dominated

→ Cosmological constant dominated

Each of these changes in different way as Universe expands - distinct expansion laws