Motion under gravity

Motions of the stars and gas in the disk of a spiral galaxy are approximately circular ($v_R$ and $v_z \ll v_\phi$).

Define the circular velocity at radius $r$ in the galaxy as $V(r)$. Acceleration of the star moving in a circular orbit must be provided by a net inward gravitational acceleration:

$$\frac{V^2(r)}{r} = \Box a_r(r)$$

To calculate $a_r(r)$, must in principle sum up gravitational force from bulge, disk and halo.
For spherically symmetric mass distributions:

- Gravitational force at radius $r$ due to matter interior to that radius is the same as if all the mass were at the center.
- Gravitational force due to matter outside is zero.

Thus, if the mass enclosed within radius $r$ is $M(r)$, gravitational acceleration is:

$$a_r = \frac{-GM(r)}{r^2}$$

(minus sign reflecting that force is directed inward)
Bulge and halo components of the Galaxy are at least approximately spherically symmetric - assume for now that those dominate the potential.

Self-gravity due to the disk itself is not spherically symmetric…

If you are familiar with vector calculus, *Sparke & Gallagher 3.1* derives Poisson’s equation needed to calculate force from an arbitrary mass distribution.

Note: no simple form for the force from disks with realistic surface density profiles…
Rotation curves of simple systems

1. Point mass M:

\[ V(r) = \sqrt{\frac{GM}{r}} \]

Applications:
- Close to the central black hole \((r < 0.1 \text{ pc})\)
- `Sufficiently far out’ that \(r\) encloses all the Galaxy’s mass

eg image of the Galactic center

Note: non-circular orbits and presence of massive stars

Movie: Andrea Ghez’s group
2. Uniform sphere:

If the density $\rho$ is constant, then:

$$M(r) = \frac{4}{3} \rho r^3$$

$$V(r) = \sqrt{\frac{4\pi G \rho}{3}} r$$

Rotation curve rises linearly with radius, period of the orbit $2\pi r / V(r)$ is a constant independent of radius.

Roughly appropriate for central regions of spiral galaxies.
3. Power law density profile:

If the density falls off as a power law:

\[ \rho(r) = \frac{\rho_0}{r_0^a} \]

…with \( a < 3 \) a constant, then:

\[ V(r) = \sqrt{\frac{4\pi G \rho_0 r_0^a}{3}} r^{1-a/2} \]

For many galaxies, circular speed curves are approximately flat \((V(r) = \text{constant})\). Suggests that mass density in these galaxies may be proportional to \( r^{-2} \).
4. Simple model for a galaxy with a core:

Spherical density distribution:

\[ 4\pi G \rho(r) = \frac{V_H^2}{r^2 + a_H^2} \]

- Density tends to constant at small \( r \)
- Density tends to \( r^{-2} \) at large \( r \)

Corresponding circular velocity curve is:

\[ V(r) = V_H \sqrt{1 - \frac{a_H}{r} \arctan \frac{r}{a_H}} \]
Resulting rotation curve