White dwarfs

End state of stars with masses M < 8 M_{sun} is a white dwarf: stellar remnant in which the pressure is provided by degenerate electrons and there is no significant nuclear burning.

Very well described as **polytropes**. Basic equations:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$ conservation of mass

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$ hydrostatic equilibrium

$$P = K\rho^{\gamma}$$ polytropic equation of state (depends only upon density)

For non-relativistic electrons, $\gamma = 5/3$. This corresponds to a **polytropic index** n = 3/2, where (by definition):

$$\gamma = 1 + \frac{1}{n}$$

Step 1: combine the two relevant stellar structure equations

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

start with hydrostatic equilibrium

$$\frac{r^2}{\rho}\frac{dP}{dr} = -Gm$$

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr}$$

Substitute for right-hand-side using the mass equation:

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\left(4\pi r^2\rho\right)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

Step 2: make use of the polytropic equation of state to eliminate pressure from the equation

$$P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}}$$

$$\frac{dP}{dr} = K\left(\frac{n+1}{n}\right)\rho^{1/n}\frac{d\rho}{dr}$$

Substitute this into the combined equation to get:

$$\frac{(n+1)K}{4\pi G n}\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho^{\frac{n-1}{n}}}\frac{d\rho}{dr}\right) = -\rho$$

Looks messy, but note that this equation **only involves** the unknown function ρ(r) - everything else is a known constant.

Step 3: introduce a dimensionless variable for the density

Define a new variable $\theta(r)$ that is related to the density via the definition:

$$\rho = \rho_c \theta^n$$

…where ρ_c is the central density of the star. θ is a dimensionless quantity, and varies between $\theta = 1$ (at $r = 0$) and $\theta = 0$ (at $r = R$, the stellar radius)

This substitution further simplifies the equation:

$$\left[\underbrace{\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}}_{\text{constants}} \right] \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\theta}{dr} \right) = -\theta^n$$

At this point, possible to reduce the whole equation to a dimensionless form by replacing r with a dimensionless variable (see Section 5.3 in the textbook). Can derive the mass - radius relation without doing this:

Stellar mass: $M = \int_0^R 4\pi r^2 \rho \, dr$

Having made the previous substitutions, this integral is now trivial:

$$\left[\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}\right] \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) = -\theta^n = -\frac{\rho}{\rho_c}$$

$$M = -4\pi \rho_c \left[\frac{(n+1)K}{4\pi G \rho_c^{\frac{n-1}{n}}}\right] \int_0^R \frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) dr$$

$$M = -\left(\frac{(n+1)K\rho_c^{1/n}}{G}\right)\int_0^R \frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right)dr$$

$$= -\frac{(n+1)K\rho_c^{1/n}}{G}\left[r^2\frac{d\theta}{dr}\right]_0^R$$

$$= -\frac{(n+1)K\rho_c^{1/n}}{G}\times R^2\left.\frac{d\theta}{dr}\right|_{r=R}$$

Since θ is dimensionless, the value of $d\theta$ / dr at r = R is proportional to R^{-1} for a fixed value of n.

$$M \propto \rho_c^{1/n} R$$

Substitute for the central density: $\rho_c \propto \frac{M}{R^3}$

Mass - radius relation for polytropes

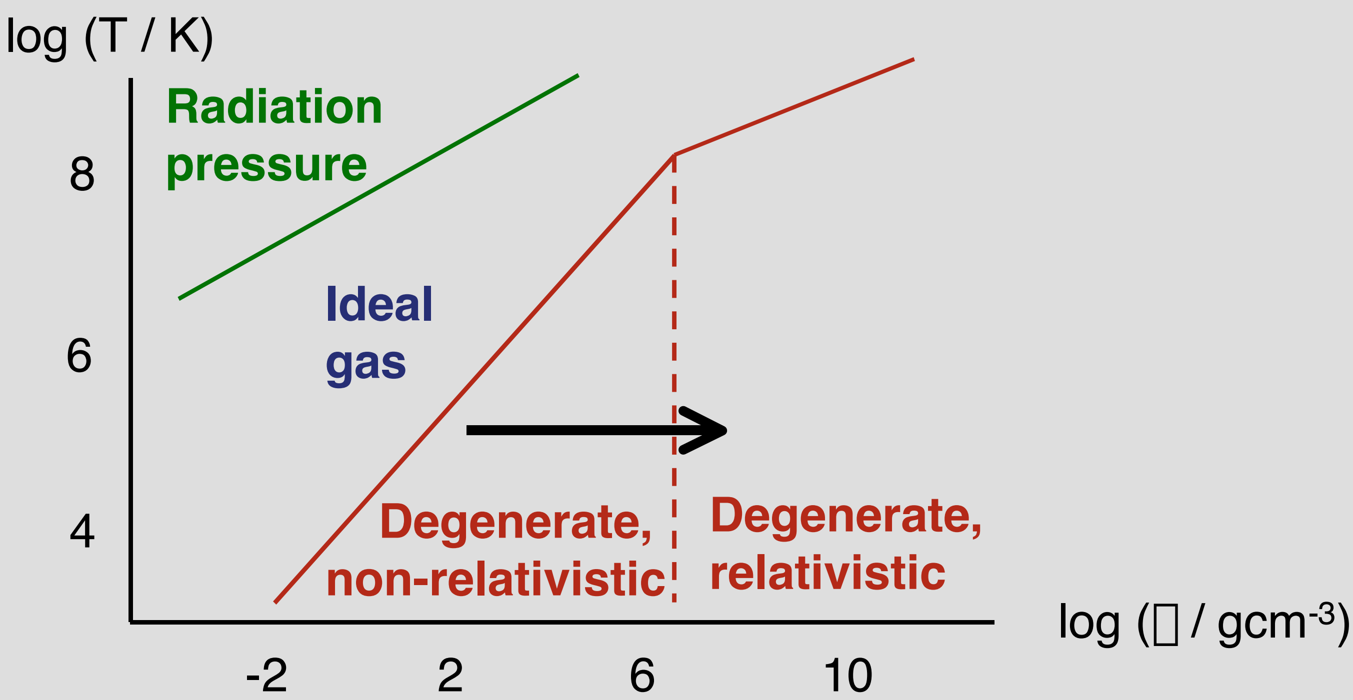
$$R \propto M^{\frac{n-1}{n-3}}$$

Application to white dwarfs

Non-relativistic degeneracy pressure: n = 3/2

$$R \propto M^{-1/3}$$

More massive white dwarfs have smaller radii. Central density increases with mass, toward the point where the most energetic electrons have relativistic velocities:



Relativistic degeneracy pressure: n = 3

$$M \quad R^{\frac{n-3}{n-1}} = \text{constant}$$

The mass of an n = 3 polytrope is not a free parameter, it is uniquely specified by the constant K in the equation of state.

K depends upon the composition. For a white dwarf made of carbon and oxygen the mass is:

$$M_{Ch} = 1.46 M_{sun}$$

Chandrasekhar mass - the maximum mass possible for a white dwarf star.

Essentially the same maximum mass applies to the degenerate core of a star whatever the composition: M_{Ch} depends upon the number of nucleons per electron which is ~2 for all elements heavier than hydrogen.

Endpoints of stellar evolution

1) Stars with $M < 8\ M_{sun}$:

These stars never develop a degenerate core more massive than the Chandrasekhar limit (for the more massive stars, this requires a lot of mass loss).

Endpoint is a white dwarf with a mass smaller than M_{Ch}, in which the pressure is provided by non-relativistic degenerate electrons.

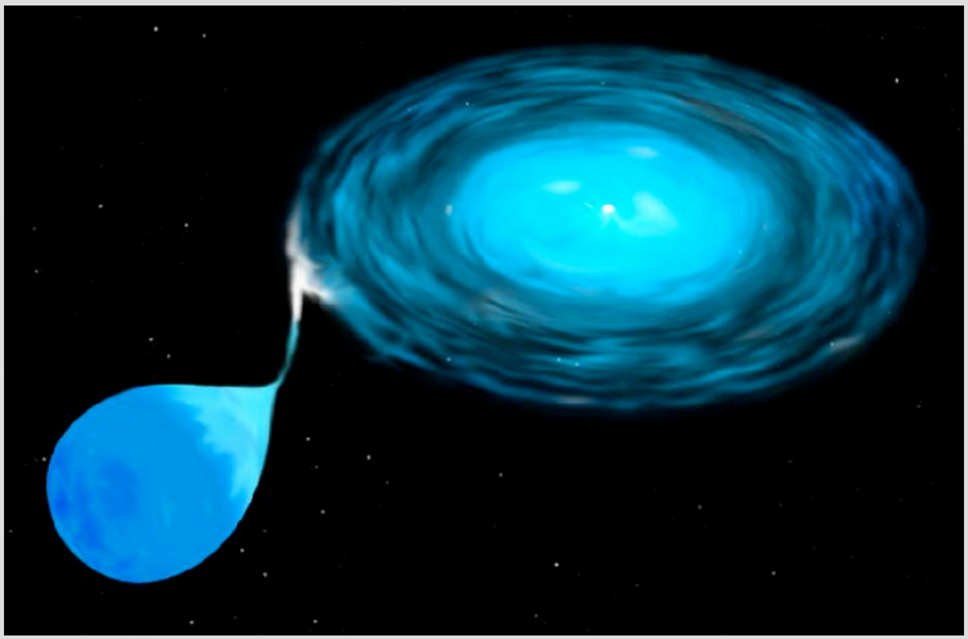
An isolated white dwarf simply cools off and becomes dimmer and dimmer for all time.

2) Stars with $M > 8\ M_{sun}$:

Nuclear reactions in these stars cease once an iron core has developed. Core is too massive to be supported by electron degeneracy, leading to **core collapse**.

Exceeding the Chandrasekhar limiting mass

Suppose we add mass to a white dwarf, for example in a mass transfer binary system, to bring it up to the Chandrasekhar limit. What happens?



Possibility 1

Once M_{Ch} is reached, the pressure of degenerate electrons can no longer hold the star up:

→ collapse

If this *accretion-induced collapse* occurs, the end state would be a neutron star. The collapse would produce very little in the way of observable phenomenon.

Possibility 2

As M approaches M_{Ch}, the temperature and density in the core ignite fresh nuclear reactions.

Unlike in the case of ordinary stellar nuclear reactions, this is devastating to the star. Recall:

$$P = K\rho^{5/3}$$ …with no temperature dependence

Hence, large energy release from nuclear reactions heats the material up without changing the pressure or density.

Reactions runaway, eventually lifting the degeneracy but not before all the star has been burned:

→ Supernova explosion, production of ~1 M_{sun} of radioactive nickel.

Observe several classes of supernova, which are classified according to the spectral lines seen in their spectra:

Type I: no lines of hydrogen in the spectrum

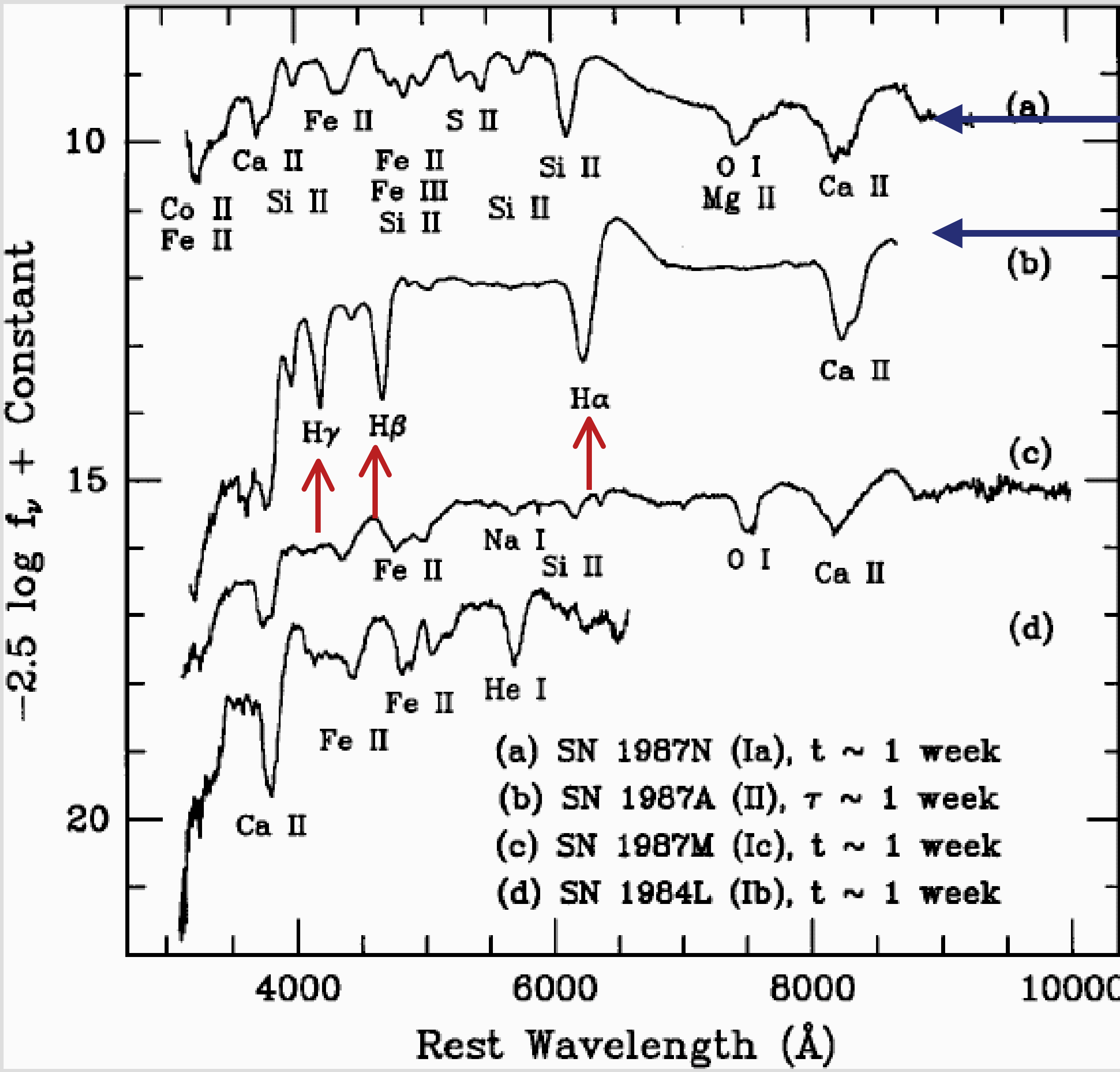
Type II: lines of hydrogen seen in spectrum

Type Is are further divided into subclasses (Ia, Ib and Ic) again based on their spectral properties.

Type Ia supernovae are believed to result from the explosion of Chandrasekar mass white dwarfs. Other types are thought to result from the collapse of massive stars.

Note: classification predates any physical understanding, and so is potentially confusing!

Spectra of different classes of supernovae



Type Ia

Type II

Clear lines of hydrogen in spectrum

Type Ia supernovae

Identification of Type Ia supernovae with exploding white dwarfs is circumstantial but strong. Main clues are:

- No H lines but presence of Si lines in absorption
  - At most ~ 0.1 Solar masses of H in vicinity
  - Nuclear burning all the way to Si must occur
- Observed in elliptical galaxies as well as spirals
  - Old stellar population - not massive stars
- Remarkably homogenous properties
  - `Same object' exploding in each case
- Lightcurve fit by radioactive decay of about a Solar mass of ^{56}Ni

Does **not** mean that accretion-induced collapse does not occur in some circumstances as well…