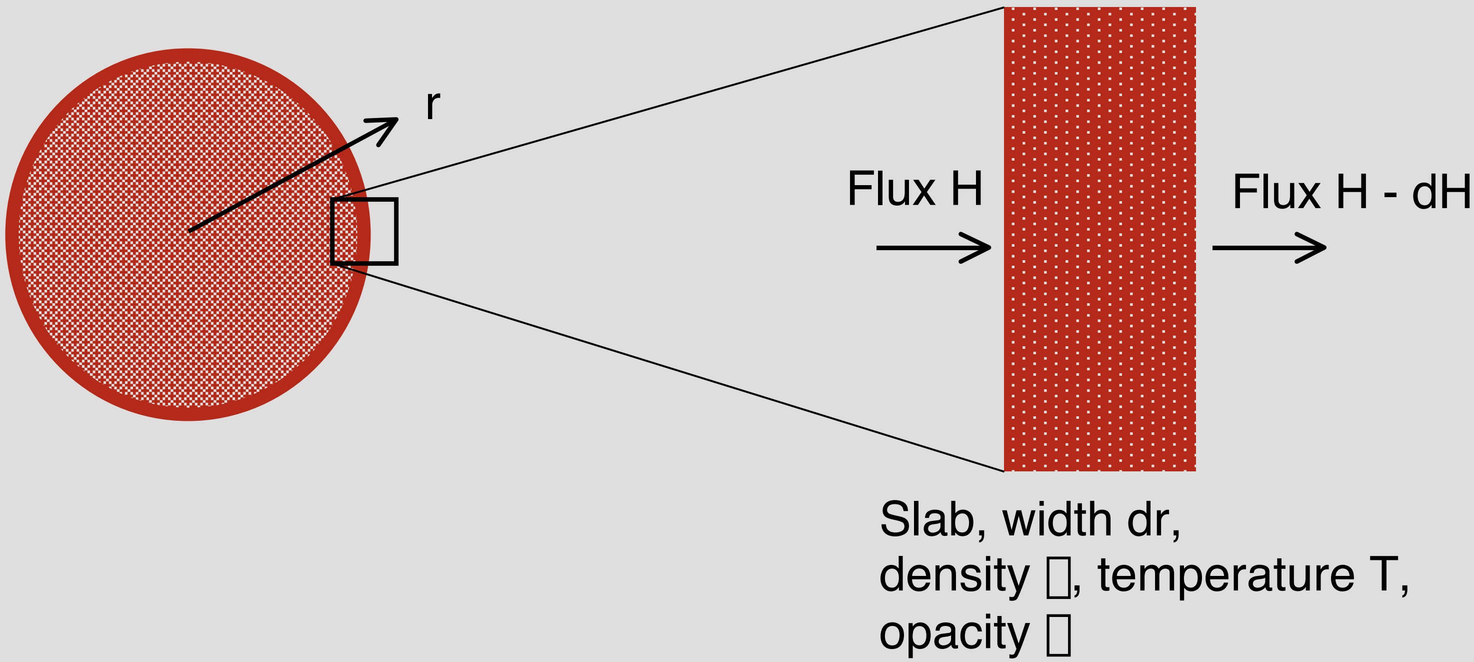
Energy transport in stars



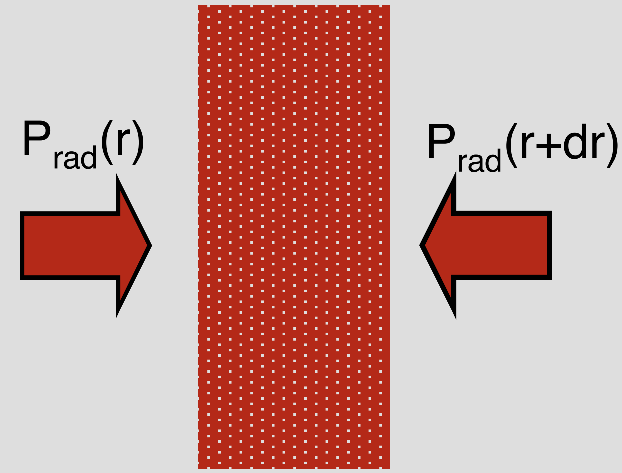
A simple derivation of the rate of energy transport due to radiation is given in Section 3.7. Consider flux of radiation (energy per unit time per unit area) passing through a slab. Change in flux:

$$\frac{dH}{H} = -\kappa\rho\, dr$$

decrease due to absorption of photons in the slab

Photons absorbed in the slab impart momentum to the gas: the momentum per unit time per unit area is:

$$\frac{|dH|}{c}$$



Rate of increase of the momentum must be equal to the net force applied to the slab by the radiation field (Newton's second law). This is equal to the *difference* in the radiation pressure between the two faces:

$$P_{rad}(r) - P_{rad}(r + dr) = -\frac{dP_{rad}}{dr} \times dr$$

$$\Rightarrow \quad \frac{|dH|}{c} = -\frac{dP_{rad}}{dr} dr$$

Substitute:

$$\frac{|dH|}{c} = -\frac{dP_{rad}}{dr}dr$$

$$|dH| = \kappa\rho H dr$$

$$P_{rad} = \frac{1}{3}aT^4 \Rightarrow \frac{dP_{rad}}{dr} = \frac{4}{3}aT^3 \frac{dT}{dr}$$

$$\frac{-\kappa\rho H dr}{c} = -\frac{4}{3}aT^3\frac{dT}{dr}dr$$

$$H = -\frac{4acT^3}{3\kappa\rho}\frac{dT}{dr}$$

If the luminosity through the entire spherical shell of radius r is L(r), then $L(r) = 4\pi r^2 H(r)$:

$$L = -\frac{16\pi r^2 acT^3}{3\kappa\rho}\frac{dT}{dr}$$

Luminosity is proportional to the temperature gradient, but with a complicated multiplying function.

Invert this to obtain the temperature gradient in terms of the luminosity at radius r:

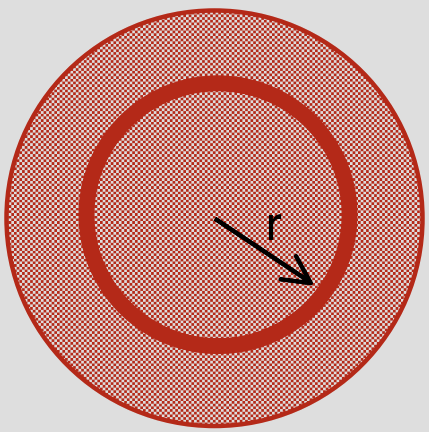
$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$

- Third equation of stellar structure, tells us what the temperature gradient is in terms of the luminosity, density and temperature.
- This derivation is rather shaky - see the textbook Appendix for a better effort.

Equation of energy generation

Assume that the star is in thermal equilibrium - i.e. at each radius the gas is neither heating up nor cooling down with time.

Let the **rate of energy generation per unit mass** be q (with units erg s^{-1} g^{-1}). Then:



Shell, mass $dm = 4\pi r^2 \rho dr$
Luminosity at r: L(r)
Luminosity at r+dr: L(r)+dL

$$dL = 4\pi r^2 \rho dr \times q$$

$$\frac{dL}{dr} = 4\pi r^2 \rho q$$

4th stellar structure equation

Summary: equations of stellar structure

At radius r in a static, spherically symmetric star:

- density ρ
- enclosed mass m (mass at *smaller radii)*
- temperature T
- luminosity L

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

Mass conservation

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

Hydrostatic equilibrium

$$\frac{dT}{dr} = -\frac{3}{4ac}\frac{\kappa\rho}{T^3}\frac{L}{4\pi r^2}$$

Energy transport due to radiation (only)

$$\frac{dL}{dr} = 4\pi r^2 \rho q$$

Energy generation

4 equations in 4 unknowns - enough for a solution once we know $P(\rho,T)$, κ and q.

Simple stellar models

In general, since κ and (especially) q are strong functions of density and temperature, these equations need to be solved numerically.

General behavior can be determined from simpler models.

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

Most useful approximation: if the pressure is **only** a function of the density then can solve these two equations independently from the equations involving temperature.

e.g. for degeneracy pressure:

$$P \propto \rho^{5/3}$$ …in non-relativistic case

So at least one important equation of state falls into this category.

Consider an equation of state of the form:

$$P = K\rho^{\gamma}$$

…where K and γ are constants. Define a **polytropic index n**, via:

$$\gamma = 1 + \frac{1}{n}$$

For example:

- Non-relativistic degeneracy: γ = 5/3, n = 1.5
- Relativistic degeneracy: γ = 4/3, n = 3

Solutions to the mass conservation and hydrostatic equilibrium equations using a polytropic equation of state are called **polytropes** - e.g. the non-relativistic degeneracy model would be a polytrope of index 1.5.

Can work out simple solutions for n=0, n=1 and n=5.