Equation of state in stars

Interior of a star contains a mixture of ions, electrons, and radiation (photons). For most stars (exception very low mass stars and stellar remnants) the ions and electrons can be treated as an ideal gas and quantum effects can be neglected.

**Total pressure:** \[ P = P_i + P_e + P_r \]

\[ = P_{\text{gas}} + P_r \]

- \( P_i \) is the pressure of the ions
- \( P_e \) is the electron pressure
- \( P_r \) is the radiation pressure
Gas pressure

The equation of state for an ideal gas is:

\[ P_{\text{gas}} = nkT \]

\[ \uparrow \]

n is the number of particles per unit volume. 
\[ n = n_i + n_e \], where \( n_i \) and \( n_e \) are the number densities of ions and electrons respectively.

In terms of the mass density \[ \Box \] :

\[ P_{\text{gas}} = \frac{\Box}{\Box m_H} \Box kT \]

…where \( m_H \) is the mass of hydrogen and \( \Box \) is the average mass of particles in units of \( m_H \). Define the ideal gas constant:

\[ R \equiv \frac{k}{m_H} \]

\[ \rightarrow \]

\[ P_{\text{gas}} = \frac{R}{\Box} \Box T \]
Determining $\mu$

$\mu$ will depend upon the composition of the gas and the state of ionization. For example:

- Neutral hydrogen: $\mu = 1$
- Fully ionized hydrogen: $\mu = 0.5$

In the central regions of stars, OK to assume that all the elements are fully ionized. Bookkeeping task to determine what $\mu$ is.

Denote abundances of different elements per unit mass by:

- $X$ hydrogen - mass $m_H$, one electron
- $Y$ helium - mass $4m_H$, two electrons
- $Z$ the rest, `metals’, average mass $A m_H$, approximately $(A / 2)$ electrons per nucleus
If the density of the plasma is \( n \), then add up number densities of hydrogen, helium, and metal nuclei, plus electrons from each species:

<table>
<thead>
<tr>
<th>Species</th>
<th>Number density of nuclei</th>
<th>Number density of electrons</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hydrogen</td>
<td>( \frac{X}{m_H} )</td>
<td>( \frac{2X}{m_H} )</td>
</tr>
<tr>
<td>Helium</td>
<td>( \frac{Y}{4m_H} )</td>
<td>( \frac{2Y}{4m_H} )</td>
</tr>
<tr>
<td>Metals</td>
<td>( \frac{Z}{Am_H} )</td>
<td>( \frac{A}{2} \frac{Z}{Am_H} )</td>
</tr>
</tbody>
</table>

\[
 n = \frac{2X}{m_H} + \frac{3}{4} Y + \frac{1}{2} Z = \frac{2X + \frac{3}{4} Y + 2Z}{m_H} \quad \text{...assuming that} \quad A \gg 1
\]
Radiation pressure

Expression for the radiation pressure of blackbody radiation is derived in Section 3.4 of the textbook. Result:

\[ P_r = \frac{1}{3} a T^4 \]

…where \( a \) is the radiation constant:

\[ a = \frac{8\pi^5 k^4}{15c^3 h^3} = \frac{4\pi}{c} \]

\[ = 7.565 \times 10^{15} \text{ erg cm}^{-3} \text{ K}^{-4} \]

\[ = 7.565 \times 10^{16} \text{ J m}^{-3} \text{ K}^{-4} \]
Conditions in the Solar core

A detailed model of the Sun gives core conditions of:

- \( T = 1.6 \times 10^7 \) K
- \( \rho = 150 \) g cm\(^{-3}\)
- \( X = 0.34, Y = 0.64, Z = 0.02 \) (note: hydrogen is almost half gone compared to initial or surface composition!)

\[ \mu = 2X + \frac{3}{4}Y + 2Z \quad \Rightarrow \quad \mu = 0.83 \]

Ideal gas constant is \( R = 8.3 \times 10^7 \) erg g\(^{-1}\) K\(^{-1}\)

Ratio of radiation pressure to gas pressure is therefore:

\[ \frac{P_r}{P_{gas}} = \frac{1}{3} \frac{aT^4}{R} = \frac{\mu a}{3R} \frac{T^3}{\mu} = 7 \times 10^4 \]

Radiation pressure is not at all important in the center of the Sun under these conditions.
In which stars are gas and radiation pressure important?

\[
\begin{align*}
    P_{\text{gas}} &= \frac{R}{a} T \\
    P_r &= \frac{1}{3} a T^4
\end{align*}
\]

Equal when:

\[
T^3 = \frac{3R}{a}
\]

\[
\log T
\]

- Gas pressure dominated
- Radiation pressure dominated

Slope 1/3
Using the virial theorem, we deduced that the characteristic temperature in a star scales with the mass and radius as:
\[
\overline{T} \propto \frac{M}{R} \quad \text{(see also textbook 2.4)}
\]

The average density scales as:
\[
\rho \propto \frac{M}{R^3}
\]

The ratio of radiation pressure to gas pressure in a star is:
\[
\frac{P_r}{P_{gas}} = \frac{1}{3} \frac{aT^4}{R} = \frac{a}{3R} \sqrt[3]{T} \sqrt[3]{\mu} \frac{M^3/R^3}{M/R^3} \mu M^2
\]

Gas pressure is most important in low mass stars
Radiation pressure is most important in high mass stars