Equation of state in stars

Interior of a star contains a mixture of ions, electrons, and radiation (photons). For most stars (exception very low mass stars and stellar remnants) the ions and electrons can be treated as an ideal gas and quantum effects can be neglected.

Total pressure:

$$P = P_I + P_e + P_r$$
$$= P_{gas} + P_r$$

- P_I is the pressure of the ions
- P_e is the electron pressure
- P_r is the radiation pressure

Gas pressure

The equation of state for an ideal gas is:

$$P_{gas} = nkT$$

n is the number of particles per unit volume.
$n = n_I + n_e$, where n_I and n_e are the number densities of ions and electrons respectively

In terms of the mass density ρ: $P_{gas} = \frac{\rho}{\mu m_H} kT$

…where m_H is the mass of hydrogen and μ is the average mass of particles in units of m_H. Define the **ideal gas constant**:

$$R \equiv \frac{k}{m_H} \quad \Rightarrow \quad P_{gas} = \frac{R}{\mu} \rho T$$

Determining μ

μ will depend upon the composition of the gas and the state of ionization. For example:

- Neutral hydrogen: $\mu = 1$
- Fully ionized hydrogen: $\mu = 0.5$

In the central regions of stars, OK to assume that all the elements are fully ionized. Bookeeping task to determine what μ is.

Denote abundances of different elements per unit mass by:

- **X** hydrogen - mass m_H, one electron
- **Y** helium - mass $4m_H$, two electrons
- **Z** the rest, `metals’, average mass Am_H, approximately (A / 2) electrons per nucleus

If the density of the plasma is ρ, then add up number densities of hydrogen, helium, and metal nuclei, plus electrons from each species:

	H	**He**	**metals**
Number density of nuclei	$\frac{X\rho}{m_H}$	$\frac{Y\rho}{4m_H}$	$\frac{Z\rho}{Am_H}$
Number density of electrons	$\frac{X\rho}{m_H}$	$\frac{2Y\rho}{4m_H}$	$\left(\frac{A}{2}\right)\frac{Z\rho}{Am_H}$

$$n = \frac{\rho}{m_H}\left(2X + \frac{3}{4}Y + \frac{1}{2}Z\right) = \frac{\rho}{\mu m_H}$$

…assuming that A >> 1

$$\mu^{-1} = 2X + \frac{3}{4}Y + 2Z$$

Radiation pressure

Expression for the radiation pressure of blackbody radiation is derived in Section 3.4 of the textbook. Result:

$$P_r = \frac{1}{3} a T^4$$

…where a is the *radiation constant*:

$$a = \frac{8\pi^5 k^4}{15 c^3 h^3} = \frac{4\sigma}{c}$$

$$= 7.565 \times 10^{-15} \text{ erg cm}^{-3} \text{ K}^{-4}$$

$$= 7.565 \times 10^{-16} \text{ J m}^{-3} \text{ K}^{-4}$$

Conditions in the Solar core

A detailed model of the Sun gives core conditions of:

- $T = 1.6 \times 10^{7}$ K
- $\rho = 150$ g cm^{-3}
- $X = 0.34$, $Y = 0.64$, $Z = 0.02$ (note: hydrogen is almost half gone compared to initial or surface composition!)

$$\mu^{-1} = 2X + \frac{3}{4}Y + 2Z \quad \Rightarrow \quad \mu = 0.83$$

Ideal gas constant is $R = 8.3 \times 10^{7}$ erg g^{-1} K^{-1}

Ratio of radiation pressure to gas pressure is therefore:

$$\frac{P_r}{P_{gas}} = \frac{\frac{1}{3}aT^4}{\frac{R}{\mu}\rho T} = \frac{\mu a}{3R} \times \frac{T^3}{\rho} = 7 \times 10^{-4}$$

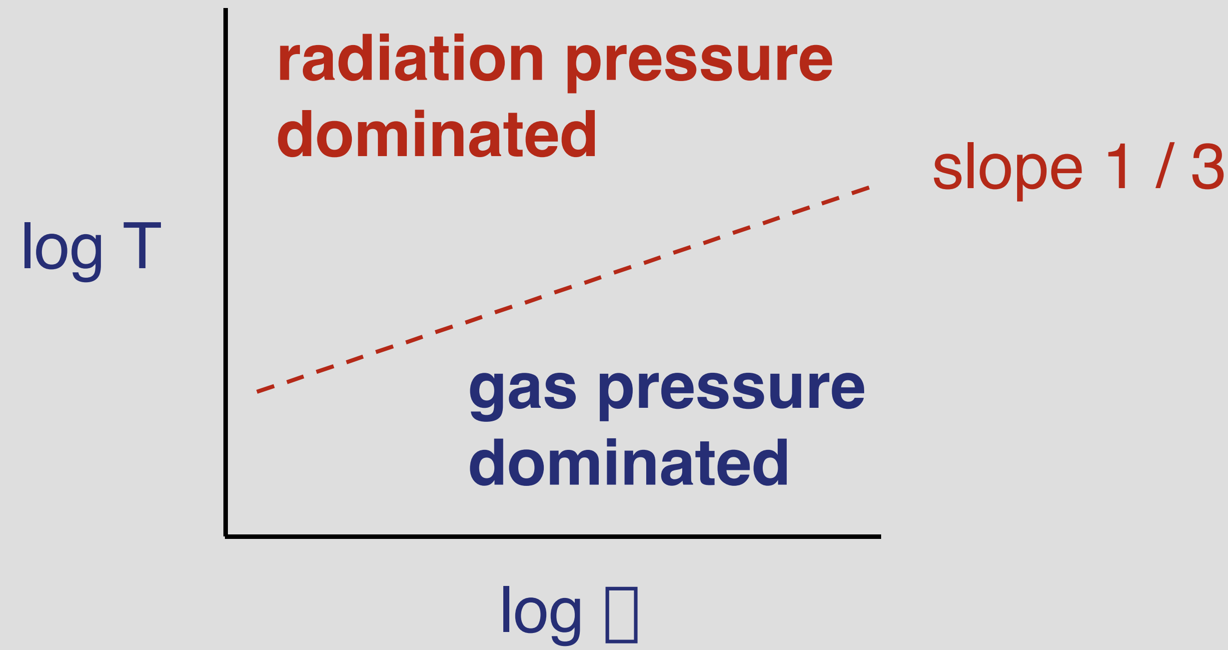
Radiation pressure is not at all important in the center of the Sun under these conditions

In which stars are gas and radiation pressure important?

$$P_{gas} = \frac{R}{\mu} \rho T$$

$$P_r = \frac{1}{3} a T^4$$

equal when: $T^3 = \frac{3R}{a\mu} \rho$



Using the virial theorem, we deduced that the characteristic temperature in a star scales with the mass and radius as:

$$\bar{T} \propto \frac{M}{R}$$ (see also textbook 2.4)

The average density scales as: $\bar{\rho} \propto \dfrac{M}{R^3}$

The *ratio* of radiation pressure to gas pressure in a star is:

$$\frac{P_r}{P_{gas}} = \frac{\frac{1}{3} a T^4}{\frac{R}{\mu} \rho T} = \frac{\mu a}{3R} \frac{T^3}{\rho} \propto \frac{M^3/R^3}{M/R^3} \propto M^2$$

Gas pressure is most important in **low mass stars**
Radiation pressure is most important in **high mass stars**