Stellar structure

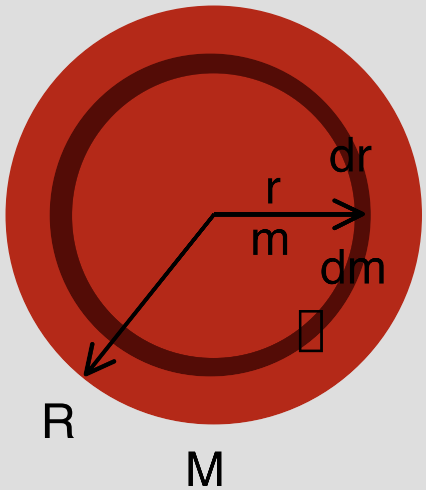
For an isolated, static, spherically symmetric star, four basic laws / equations needed to describe structure:

- **Conservation of mass**
- **Conservation of energy** (at each radius, the change in the energy flux equals the local rate of energy release)
- **Equation of hydrostatic equilibrium** (at each radius, forces due to pressure differences balance gravity)
- **Equation of energy transport** (relation between the energy flux and the local gradient of temperature)

Basic equations are supplemented by:

- **Equation of state** (pressure of a gas as a function of its density and temperature)
- **Opacity** (how transparent it is to radiation)
- **Nuclear energy generation rate** as $f(\rho,T)$

Conservation of mass



Let r be the distance from the center
Density as function of radius is $\rho(r)$

Let m be the mass *interior* to r, then conservation of mass implies that:

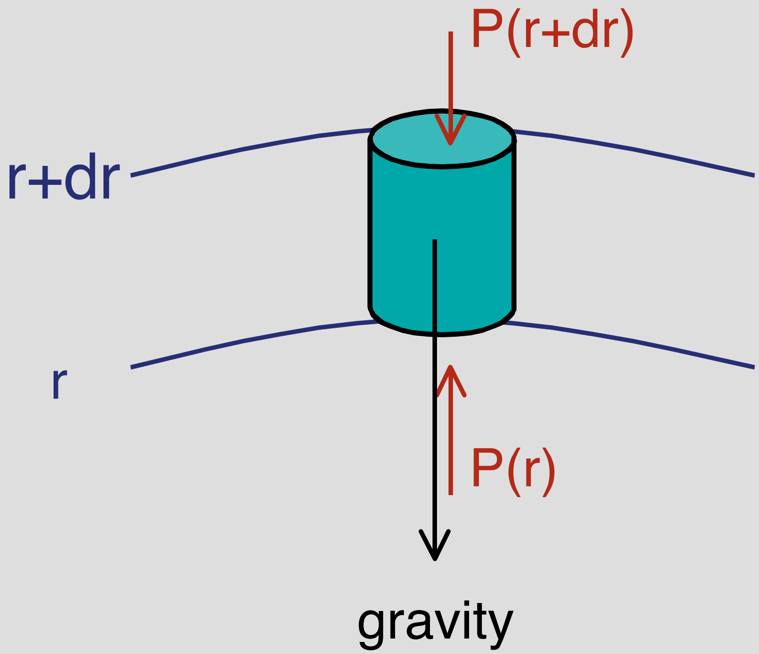
$$dm = 4\pi r^2 \rho\, dr$$

Write this as a differential equation:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

1st stellar structure equation

Equation of hydrostatic equilibrium



Consider small cylindrical element between radius r and radius r + dr in the star.

Surface area = dS
Mass = Δm

Mass of gas in the star at smaller radii = m = m(r)

Radial forces acting on the element:

Gravity (inward): $F_g = -\frac{G m \Delta m}{r^2}$

gravitational constant G = 6.67 x 10^{-8} dyne cm^2 g^{-2}

Pressure (net force due to difference in pressure between upper and lower faces):

$$F_p = P(r)dS - P(r+dr)dS$$

$$= P(r)dS - \left[P(r) + \frac{dP}{dr} dr\right] dS$$

$$= -\frac{dP}{dr} dr dS$$

Mass of element: $\Delta m = \rho dr dS$

Applying Newton's second law (`F=ma') to the cylinder:

$$\Delta m \ddot{r} = F_g + F_p = -\frac{Gm\Delta m}{r^2} - \frac{dP}{dr} dr dS$$

acceleration = 0 everywhere if star static

Setting acceleration to zero, and substituting for Δm:

$$0 = -\frac{Gm\rho\, dr\, dS}{r^2} - \frac{dP}{dr}\, dr\, dS$$

<u>**Equation of hydrostatic equilibrium**</u>:

$$\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$$

2nd stellar structure equation

If we use enclosed mass as the dependent variable, can combine these two equations into one:

$$\frac{dP}{dm} = \frac{dP}{dr} \times \frac{dr}{dm} = -\frac{Gm}{r^2}\rho \times \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

← alternate form of hydrostatic equilibrium equation

Properties of hydrostatic equilibrium equation: $\frac{dP}{dr} = -\frac{Gm}{r^2}\rho$

1) Pressure always **decreases** outward

2) Pressure gradient vanishes at r = 0

3) Condition at surface of star: P = 0 (to a good first approximation)

(2) and (3) are **boundary conditions** for the hydrostatic equilibrium equation (but why two?)

Virial theorem

Can use the hydrostatic equation to derive the virial theorem, which is very useful for estimating properties of stars:

Start with: $$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$$

Multiply both sides by volume: $V = (4 / 3)\pi r^3$:

$$VdP = \frac{4}{3}\pi r^3 \times -\frac{Gm}{4\pi r^4} dm = -\frac{1}{3}\frac{Gm}{r} dm$$

Now integrate over the whole star. LHS gives by parts:

$$\int VdP = \left[PV \right]_0^R - \int PdV$$

↑

But P = 0 at r = R, and V = 0 at r = 0, so this term vanishes

If we have a small mass dm at radius r, gravitational potential energy is:

$$d\Omega = -\frac{Gm}{r}dm$$

Hence, integrating RHS of previous equation over the star:

$$-\int \frac{1}{3}\frac{Gm}{r}dm = \frac{1}{3}\int d\Omega = \frac{1}{3}\Omega$$

where Ω is the **gravitational potential energy of the star** - i.e. the energy required to assemble the star by bringing gas from infinity (very large radius). Putting the pieces together:

$$-\int P dV = \frac{1}{3}\Omega$$

$$0 = \Omega + 3\int_0^{V(r=R)} P dV$$

← version of the virial theorem

With some assumptions about the pressure, can progress further. Often, can write the pressure in the form:

$$P = (\gamma - 1)\rho u$$

- ρ is the density
- u is the internal energy per unit mass (per gram of gas)
- γ is a constant

example: for an ideal monatomic gas $\gamma = 5 / 3$, where this is the ratio of the specific heat at constant pressure to that at constant volume.

Substitute this equation of state into the virial theorem:

$$0 = \Omega + 3 \int_0^{V(R)} (\gamma - 1)\rho u \, dV$$

ρu has units of (g cm^{-3}) x (erg g^{-1}) = erg cm^{-3} - it is the internal energy per unit **volume**

Integral of internal energy per unit volume over all volume in the star is just the total internal energy of the star, U.

$$0 = \Omega + 3(\gamma - 1)U$$

gravitational potential energy of the star

total internal energy of the star

If in some application we can estimate *either* Ω or U, can use this relation to find the other.