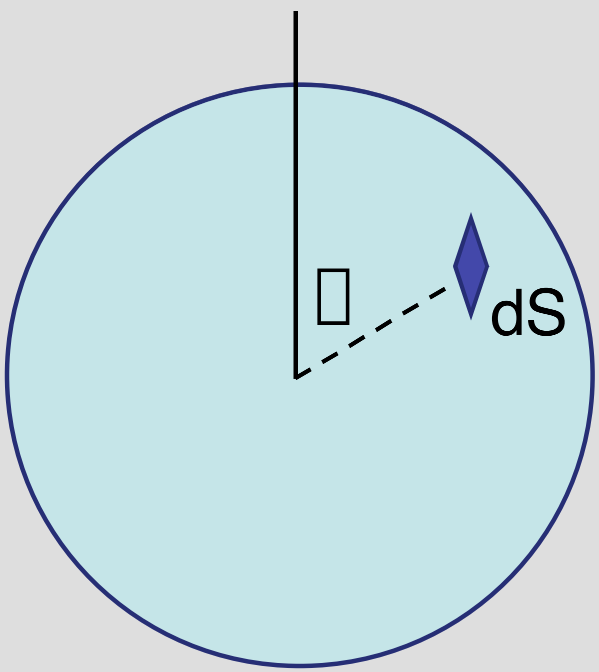
The equation of radiative transfer

How does the intensity of radiation change in the presence of emission and / or absorption?

Definition of solid angle and steradian



Sphere radius r - area of a patch dS on the surface is:

$$dS = r\,d\theta \times r\sin\theta\,d\phi \equiv r^2 d\Omega$$

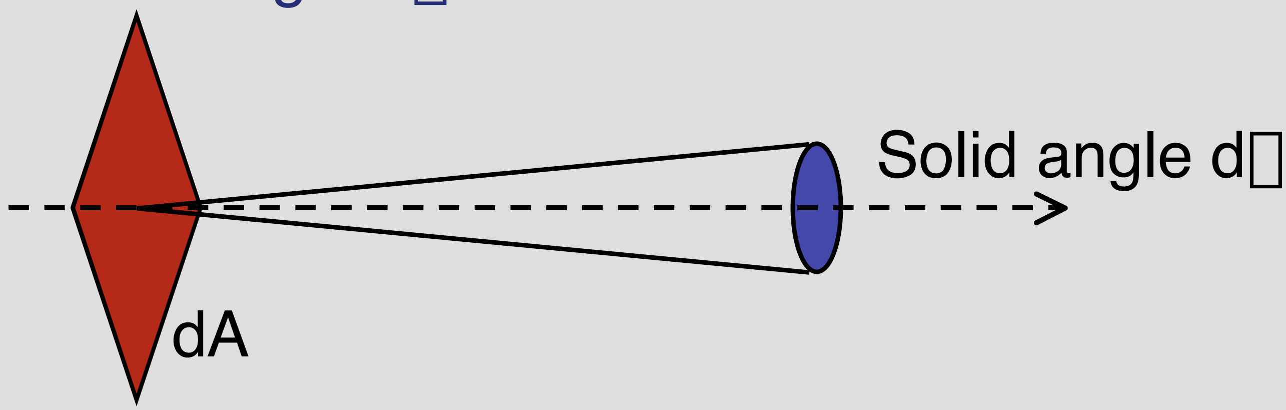
$d\Omega$ is the solid angle subtended by the area dS at the center of the sphere.

Unit of solid angle is the steradian.

4π steradians cover whole sphere.

Definition of the specific intensity

Construct an area dA normal to a light ray, and consider all the rays that pass through dA whose directions lie within a small solid angle $d\Omega$.



The amount of energy passing through dA and into $d\Omega$ in time dt in frequency range $d\nu$ is:

$$dE = I_\nu \, dA \, dt \, d\Omega \, d\nu$$

Specific intensity of the radiation.

Compare with definition of the flux: specific intensity is very similar except it depends upon direction and frequency as well as location.

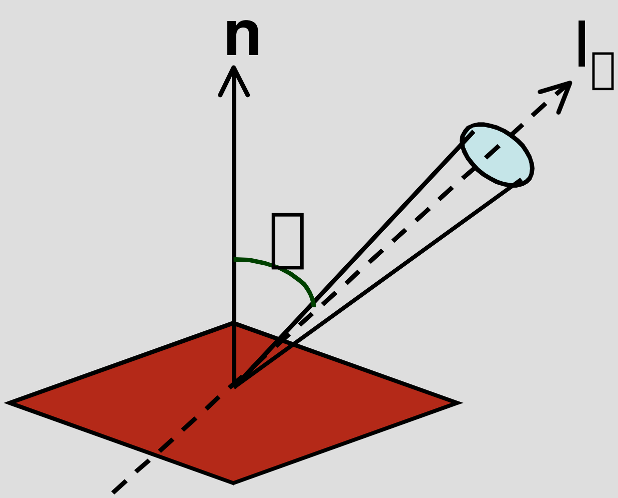
Units of specific intensity are: erg s^{-1} cm^{-2} Hz^{-1} steradian^{-1}

Same as F_{ν}

Another, more intuitive name for the specific intensity is **brightness**.

Simple relation between the flux and the specific intensity:

Consider a small area dA, with light rays passing through it at all angles to the normal to the surface **n**:



If $\theta = 90^o$, then light rays in that direction contribute **zero** net flux through area dA.

For rays at angle θ, foreshortening reduces the effective area by a factor of $\cos(\theta)$.

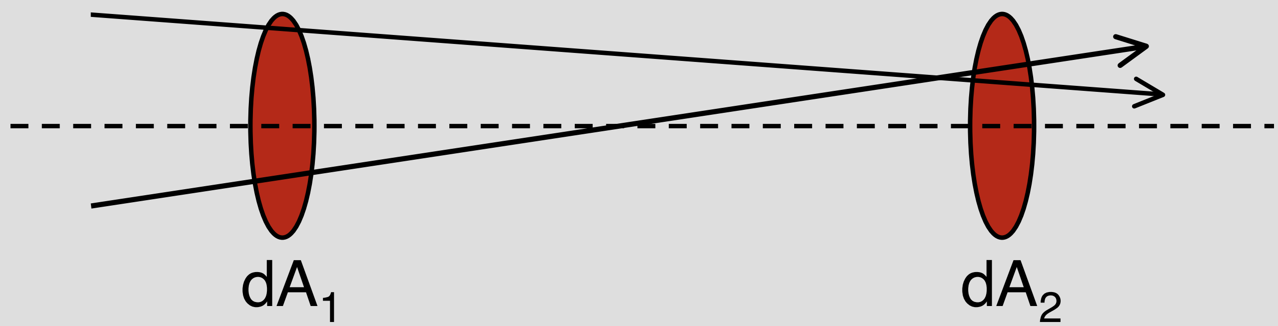
Hence, net flux in the direction of **n** is given by integrating (the specific intensity x cos θ) over all solid angles:

$$F_\nu = \int I_\nu \cos\theta d\Omega$$

Note: to actually evaluate this need to express $d\Omega$ in terms of $d\theta$ and $d\phi$ as before.

How does specific intensity change along a ray

If there is no emission or absorption, specific intensity is just **constant** along the path of a light ray. Consider any two points along a ray, and construct areas dA_1 and dA_2 normal to the ray at those points. How much energy is carried by those rays that pass through **both** dA_1 and dA_2?



$$\left.\begin{aligned} dE_1 &= I_{\nu 1} dA_1 dt d\nu_1 d\Omega_1 \\ dE_2 &= I_{\nu 2} dA_2 dt d\nu_2 d\Omega_2 \end{aligned}\right\}$$

where $d\Omega_1$ is the solid angle subtended by dA_2 at dA_1 etc

The same photons pass through both dA_1 and dA_2, without change in their frequency. Conservation of energy gives:

$$dE_1 = dE_2 \quad \text{- equal energy}$$

$$d\nu_1 = d\nu_2 \quad \text{- same frequency interval}$$

Using definition of solid angle, if dA_1 is separated from dA_2 by distance r:

$$d\Omega_1 = \frac{dA_2}{r^2}, \quad d\Omega_2 = \frac{dA_1}{r^2}$$

Substitute:

$$I_{\nu 1} dA_1 dt d\nu_1 d\Omega_1 = I_{\nu 2} dA_2 dt d\nu_2 d\Omega_2 \qquad dE_1 = dE_2$$

$$I_{\nu 1} dA_1 dt d\nu_1 \frac{dA_2}{r^2} = I_{\nu 2} dA_2 dt d\nu_2 \frac{dA_1}{r^2} \qquad d\nu_1 = d\nu_2$$

$$\underline{I_{\nu 1} = I_{\nu 2}}$$

Conclude: specific intensity remains the same as radiation propagates through free space.

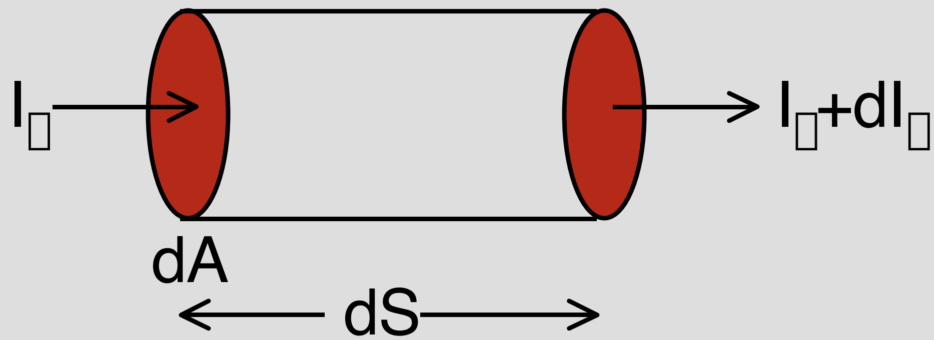
Justifies use of alternative term `brightness' - e.g. brightness of the disk of a star remains same no matter the distance - **flux** goes down but this is compensated by the light coming from a smaller area.

If we measure the distance along a ray by variable s, can express result equivalently in differential form:

$$\frac{dI_\nu}{ds} = 0$$

Emission

If the radiation travels through a medium which is itself emitting radiation, that will add to the energy:



Spontaneous **emission coefficient** is the amount of energy emitted per unit time, per unit solid angle, per unit frequency interval, and per unit volume:

$$dE = j_\nu dV d\Omega \, dt d\nu$$

In going a distance ds, beam of cross-section dA travels through a volume dV = dA x ds.

Change (increase) in specific intensity is therefore:

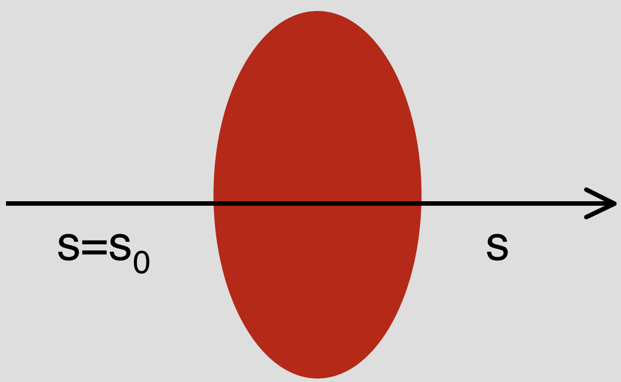
$$dI_\nu = j_\nu ds$$

Equation of radiative transfer for pure emission becomes:

$$\frac{dI_\nu}{ds} = j_\nu$$

If we know what j_ν is, can integrate this equation to find the change in specific intensity as radiation propagates through the gas:

$$I_\nu(s) = I_\nu(s_0) + \int_{s_0}^{s} j_\nu(s')\,ds'$$



i.e. add up the contributions to the emission all along the path.