Useful states and entanglement distillation, and a toy channel exhibiting superadditivity of coherent information

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Based on **arXiv:1701.03081** with Nilanjana Datta and Graeme Smith

(to appear in IEEE Transactions on Information Theory),

and **work in preparation** with Debbie Leung and Graeme Smith

QuSoft seminar, Amsterdam, 12 January 2018

Outline

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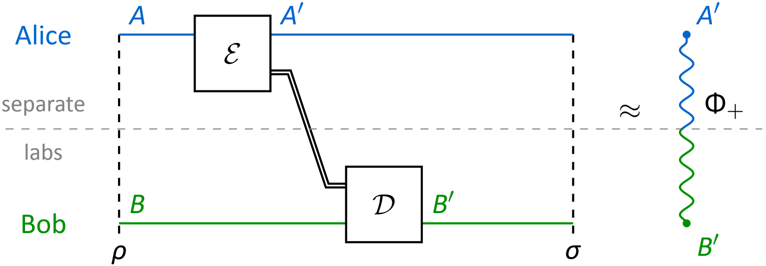
Entanglement distillation

- Entanglement can be used as a resource in: teleportation, dense coding, entanglement-assisted communication, ...
- Above tasks are easier to perform with **clean entanglement** in the form of **ebits** $|\Phi\rangle \sim |00\rangle + |11\rangle$.
- More realistic scenario: Alice and Bob share **noisy entanglement**, i.e., some mixed bipartite state ρ_{AB}.
- **Entanglement distillation:** Convert noisy entanglement into clean entanglement using local operations (LO) and classical communication (CC).
- This talk: restrict to forward or **one-way LOCC**.

Entanglement distillation using 1-LOCC



- **Alice** and **Bob** share n copies of a bipartite (mixed) state ρ_{AB}.
- **Goal:** Distill m_n copies of an ebit $|\Phi_+\rangle \sim |00\rangle + |11\rangle$.
- **Distillable entanglement** $D_\rightarrow(\rho_{AB})$: highest rate at which ebits can be distilled from ρ_{AB} with vanishing error.

Distillable entanglement: Hashing and coding theorem

- **Hashing bound**: [Devetak and Winter 2005]

$$D_{\rightarrow}(\rho_{AB}) \geq I(A\rangle B)_\rho,$$

where $I(A\rangle B)_\rho = S(B)_\rho - S(AB)_\rho$ is the coherent information.

- Define the following "single-letter" quantity:

$$D^{(1)}_{\rightarrow}(\rho_{AB}) := \sup_{\substack{\Lambda:\ AB \rightarrow A'B' \\ \text{1-LOCC}}} I(A'\rangle B')_{\Lambda(\rho)}.$$

- **Coding theorem**: [Devetak and Winter 2005]

$$D_{\rightarrow}(\rho_{AB}) = \lim_{n\rightarrow\infty} \frac{1}{n} D^{(1)}_{\rightarrow}(\rho_{AB}^{\otimes n})$$

- Regularization renders **computation** of distillable entanglement **intractable** in most cases!

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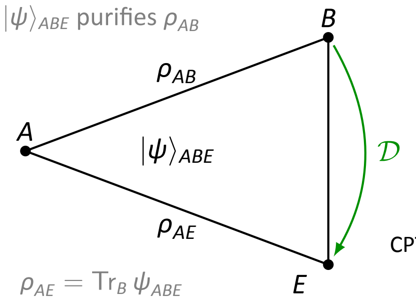
Useful and useless states for 1-LOCC

- **Hashing bound**: $D_{\rightarrow}(\rho_{AB}) \geq I(A\rangle B)$.
- Are there states for which this is optimal?

  $\longrightarrow$ **degradable states** [Devetak and Shor 2005; Smith et al. 2008]



degradable:

$\exists \mathcal{D}\colon B \rightarrow E$ s.t.

$\rho_{AE} = (\mathrm{id}_A \otimes \mathcal{D})(\rho_{AB})$

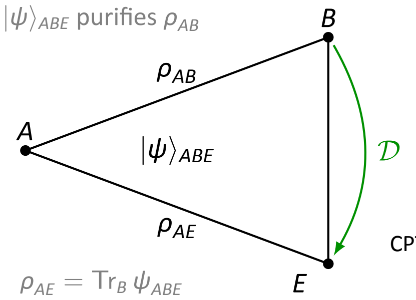
Quantum channel $\mathcal{D}\colon B \rightarrow E$:

CPTP map from op.s on $\mathcal{H}_B$ to op.s on $\mathcal{H}_E$.

Useful and useless states for 1-LOCC

- Degradable states: $D^{(1)}_{\to}(\rho_{AB}) = \sup_{\Lambda} I(A'\rangle B')_{\Lambda(\rho)} = I(A\rangle B)_{\rho}$
- Coherent information is additive: $D^{(1)}_{\to}(\rho_{AB}^{\otimes n}) = n\, I(A\rangle B)_{\rho}$.
- **Single-letter one-way distillable entanglement:**

$$D_{\to}(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} D^{(1)}_{\to}(\rho_{AB}^{\otimes n}) = I(A\rangle B)_{\rho}.$$



degradable:

$\exists \mathcal{D} \colon B \to E$ s.t.

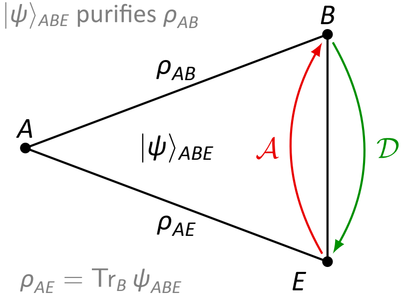
$\rho_{AE} = (\mathrm{id}_A \otimes \mathcal{D})(\rho_{AB})$

Quantum channel $\mathcal{D} \colon B \to E$:

CPTP map from op.s on $\mathcal{H}_B$ to op.s on $\mathcal{H}_E$.

Useful and useless states for 1-LOCC

- Which states are useless? $\longrightarrow$ **antidegradable states**
- These states always have $I(A\rangle B)_\rho \leq 0$ and $D^{(1)}_{\rightarrow}(\rho_{AB}) = 0$.
- Antidegradable states are **undistillable:** $D_{\rightarrow}(\rho_{AB}) = 0$ (can also be derived from no-cloning argument).



degradable:

$\exists \mathcal{D}\colon B \to E$ s.t.

$\rho_{AE} = (\mathrm{id}_A \otimes \mathcal{D})(\rho_{AB})$

antidegradable:

$\exists \mathcal{A}\colon E \to B$ s.t.

$\rho_{AB} = (\mathrm{id}_A \otimes \mathcal{A})(\rho_{AE})$

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Bounding the distillable entanglement

- **Crucial observation**: cf. [Wolf and Pérez-García 2007]

  The regularized quantity $D_{\rightarrow}(\cdot)$ is **convex on mixtures** of states with **additive** $D^{(1)}_{\rightarrow}(\cdot)$.

- Candidates:
  - Useful (degradable) states: $D^{(1)}_{\rightarrow}(\omega_{AB}) = I(A\rangle B)_{\omega} \longrightarrow$ additive.
  - Useless (antidegradable) states: $D^{(1)}_{\rightarrow}(\tau_{AB}) = 0 \longrightarrow$ additive.

Main result: Upper bound on distillable entanglement

Let $\rho_{AB} = \sum_i p_i \omega_i + \sum_i q_i \tau_i$, where the ω_i are **degradable** and the τ_i are **antidegradable**. Then,

$$D_{\rightarrow}(\rho_{AB}) \leq \sum_i p_i \, I(A\rangle B)_{\omega_i}.$$

Finding good decompositions

- **Caution:** Do such decompositions always exist?
- Yes, since pure states are always degradable (easy to see).
- Hence, every **pure-state decomposition** of ρ_{AB} is a **feasible point** for upper bound.
- Optimum for these: **entanglement of formation**

$$E_F(\rho_{AB}) := \inf_{\{p_x, |\psi^x\rangle\}} \sum_x p_x \, S(\mathrm{Tr}_B \, \psi^x_{AB}),$$

where infimum is over $\{p_x, |\psi^x\rangle_{AB}\}$ s.t. $\rho_{AB} = \sum_x p_x \psi^x_{AB}$.

- Hence, $D_{\to}(\rho_{AB}) \leq \sum_i p_i \, I(A\rangle B)_{\omega_i} \leq E_F(\rho_{AB})$.
- **Challenge**: Find decompositions into **mixed states** (more later).

Upper bound for two-way distillable entanglement

- We can similarly define the **two-way distillable entanglement** $D_{\leftrightarrow}(\rho_{AB})$ as the maximum rate at which we can distill ebits from ρ_{AB} using a **two-way LOCC operation**.

- $D_{\leftrightarrow}(\rho_{AB})$ is also given by regularized formula and **intractable to compute** in general. [Devetak and Winter 2005]

- Our method can be carried over to the two-way setting:

  - Useful states are **maximally correlated states** of the form $\sum_{ij} M_{ij} |ii\rangle\langle jj|$, since $D_{\leftrightarrow}(\rho_A B) = I(A\rangle B)_\rho = I(B\rangle A)_\rho$. [Rains 1999]

  - Useless states are the PPT states. [Horodecki et al. 1998]

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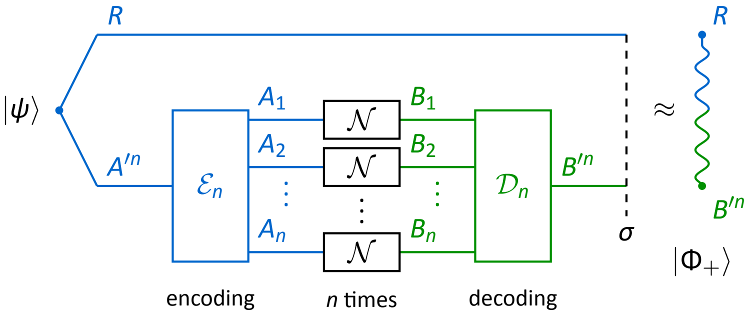
Entanglement generation through quantum channels



- **Goal:** Generate m_n ebits $|\Phi_+\rangle \sim |00\rangle + |11\rangle$ through n uses of the quantum channel $\mathcal{N}$.
- **Alice** prepares $|\psi\rangle_{RA'^n}$ and sends A'^n to **Bob** through $\mathcal{N}^{\otimes n}$.
- **Quantum capacity** $Q(\mathcal{N})$: largest possible rate at which ebits can be generated with vanishing error.

Quantum capacity

- **LSD theorem**: [Lloyd 1997; Shor 2002; Devetak 2005]

$$Q(\mathcal{N}) = \lim_{n\to\infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n}) \qquad (*)$$

  where the **channel coherent information** $Q^{(1)}(\mathcal{N})$ is defined as

$$Q^{(1)}(\mathcal{N}) := \max_{|\psi\rangle_{A'A}} I(A'\rangle B)_{(\mathrm{id}\otimes\mathcal{N})(\psi)}.$$

- **Regularized formula** $(*)$ suffers from same problem as in entanglement distillation: in general **intractable to compute**.

- Notorious example: Qubit depolarizing channel

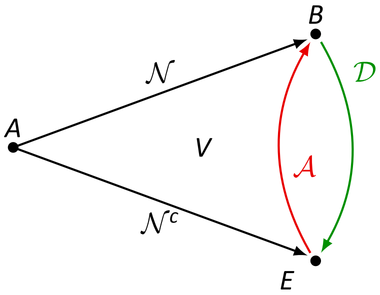
$$\mathcal{D}_p(\rho) := (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z).$$

Qubit depolarizing channel

- **Known:** $Q(\mathcal{D}_0) = 1$ and $Q(\mathcal{D}_p) = 0$ for $p \geq 0.25$ (no-cloning).
- **Unknown:** $Q(\mathcal{D}_p)$ for $p \in (0, 1/4)$.
- Partial answer for *low noise* ($p \gtrsim 0$): [FL, Leung, Smith 2017]
  $Q(\mathcal{D}_p) \approx Q^{(1)}(\mathcal{D}_p) + O(p^2 \log p)$ based on [Sutter et al. 2017]
- **Superadditivity:** $Q^{(1)}(\mathcal{D}_p) = 0$ for $p \geq 0.1894$, but
  $Q^{(1)}(\mathcal{D}_p^{\otimes 3}) > 0$ for $p \lesssim 0.1901$. [DiVincenzo et al. 1998]
- Achieved by **highly degenerate** repetition code $\sim |0\rangle^{\otimes n} + |1\rangle^{\otimes n}$.
- Largest threshold ($p \approx 0.1904$) for $n = 5$ uses of $\mathcal{D}_p$.
- Result: we can have $Q^{(1)}(\mathcal{N}^{\otimes n}) > nQ^{(1)}(\mathcal{N})$.

Degradable and antidegradable channels

- For every $\mathcal{N}: A \to B$ there is a (non-unique) **isometric extension** $V: A \to B \otimes E$ with $\mathcal{N}(\rho) = \mathrm{Tr}_E(V\rho V^\dagger)$. [Stinespring 1955]
- The **complementary channel** $\mathcal{N}^c: A \to E$ associated to $\mathcal{N}$ (and V) is defined as $\mathcal{N}^c(\rho) := \mathrm{Tr}_B(V\rho V^\dagger)$.
- $\mathcal{N}^c$ models the **leakage of information** to the environment.

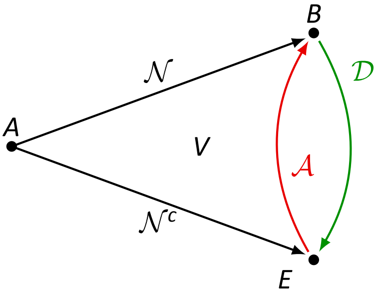


degradable:
$\exists \mathcal{D}: B \to E$ s.t.
$\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$

antidegradable:
$\exists \mathcal{A}: E \to B$ s.t.
$\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$

Degradable and antidegradable channels

- Degradable channels have additive channel coherent information, $Q^{(1)}(\mathcal{N}^{\otimes n}) = nQ^{(1)}(\mathcal{N})$. [Devetak and Shor 2005]
- **Single-letter quantum capacity**: $Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$.
- Antidegradable channels: $Q(\mathcal{N}) = 0$ due to **no-cloning**.
- For example, $\mathcal{D}_p$ is antidegradable for $p \geq 1/4$.



degradable:
$\exists \mathcal{D}: B \to E$ s.t.
$\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$

antidegradable:
$\exists \mathcal{A}: E \to B$ s.t.
$\mathcal{N} = \mathcal{A} \circ \mathcal{N}^c$

Bounding the quantum capacity

- Original statement from [Wolf and Pérez-García 2007]:
  $Q(\cdot)$ is **convex** on channels with **additive** $Q^{(1)}(\cdot)$.
  (We used this for $D_{\rightarrow}(\cdot)$ and $D^{(1)}_{\rightarrow}(\cdot)$ before).
- This is true even if the channels in a decomposition are only completely positive, but not necessarily trace-preserving.

Upper bound on quantum capacity (see also [Yang, TBP])

Let $\mathcal{N} = \sum_i p_i \mathcal{E}_i + \sum_i q_i \mathcal{F}_i$, where the $\mathcal{E}_i$ are **degradable** CP maps and the $\mathcal{F}_i$ are **antidegradable**. Then,

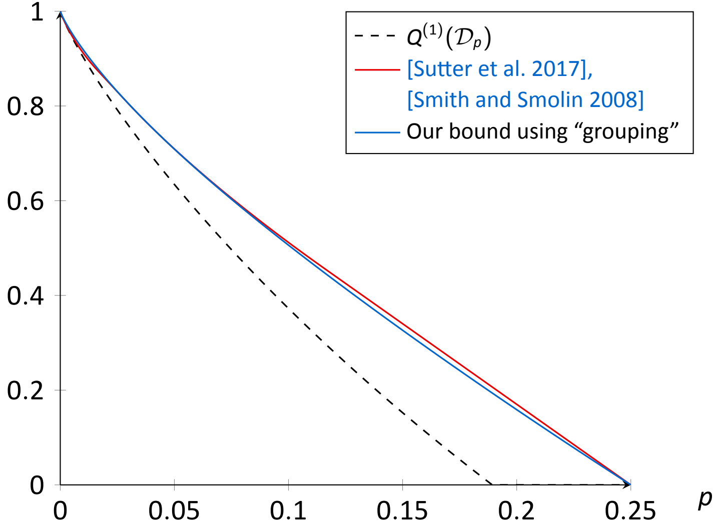
$$Q(\mathcal{N}) \leq \sum_i p_i \, Q^{(1)}(\mathcal{E}_i).$$

- How can we use this to bound $Q(\mathcal{D}_p)$?

Bounding the quantum capacity of $\mathcal{D}_p$

- Every qubit-qubit map with two Kraus operators ($|E| = 2$) is either deg. or antideg. [Wolf and Pérez-García 2007]
- **Grouping recipe:** ($\psi_i \equiv |\psi_i\rangle\langle\psi_i|, \mathcal{J}(\mathcal{N}) \ldots$ Choi state of $\mathcal{N}$)
  - Take a pure-state decomposition
    $$\mathcal{J}(\mathcal{D}_p) = p_1\,\psi_1 + p_2\,\psi_2 + p_3\,\psi_3 + p_4\,\psi_4 + \ldots \qquad (*)$$
  - $\omega_1 := \frac{p_1}{p_1+p_2}\psi_1 + \frac{p_2}{p_1+p_2}\psi_2, \quad \omega_2 := \frac{p_3}{p_3+p_4}\psi_3 + \frac{p_4}{p_3+p_4}\psi_4, \ldots$
  - Then, rank $\omega_i = 2$ and $\rho_{AB} = (p_1+p_2)\omega_1 + (p_3+p_4)\omega_2 + \ldots$
  - ω_i are Choi states of qubit-qubit CP maps with $|E| = 2$.
  - Apply our main result and optimize over decompositions $(*)$ (which are indexed by unitaries [Hughston et al. 1993]).
- This recipe yields best known bound on $\mathcal{D}_p$ in high-noise regime!

Upper and lower bounds on $Q(\mathcal{D}_p)$



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Optimality of our bound

Main principle

For a channel $\mathcal{N} = (1-\lambda)\mathcal{E} + \lambda\mathcal{F}$, with $\mathcal{E}$ degradable and $\mathcal{F}$ antidegradable, we only count **degradable contributions**:

$$Q(\mathcal{N}) \leq (1-\lambda)Q^{(1)}(\mathcal{E}) = (1-\lambda)Q^{(1)}(\mathcal{E}) + \lambda \underbrace{Q^{(1)}(\mathcal{F})}_{=0} =: E_{DA}(\mathcal{N})$$

- Is there hope to improve our bound by also counting **(negative) antidegradable contributions**?
- Good candidate:

$$E_{DA}(\mathcal{N}) = (1-\lambda) \max_{\varphi} I(A\rangle B)_{\mathcal{E}(\varphi)} + \lambda \max_{\varphi} I(A\rangle B)_{\mathcal{F}(\varphi)}$$

$$\geq \max_{\varphi} \left\{ (1-\lambda) I(A\rangle B)_{\mathcal{E}(\varphi)} + \lambda \underbrace{I(A\rangle B)_{\mathcal{F}(\varphi)}}_{\leq 0} \right\} \overset{?}{\geq} Q(\mathcal{N})$$

Optimality of our bound

- Simple case: **flagged channel** (here, both $\mathcal{E}$ and $\mathcal{F}$ are TP)

$$\mathcal{N}_f = (1-\lambda)\mathcal{E} \otimes |0\rangle\langle 0| + \lambda \mathcal{F} \otimes |1\rangle\langle 1| \qquad (*).$$

- Bob can decide which channel occurred by first measuring flag.
- Easy to show:

$$Q^{(1)}(\mathcal{N}_f) = \max_\varphi \left\{ (1-\lambda) I(A\rangle B)_{\mathcal{E}(\varphi)} + \lambda I(A\rangle B)_{\mathcal{F}(\varphi)} \right\}$$

- This is the conjectured upper bound!
- Hence, if true, any channel $\mathcal{N}_f$ of the form $(*)$ must have **additive coherent information**, since

$$Q^{(1)}(\mathcal{N}_f) \leq Q(\mathcal{N}_f) \leq \text{ upper bound } = Q^{(1)}(\mathcal{N}_f).$$

- **Counterexample!** to appear: [FL, Leung, Smith 2018]

Dephasing and erasure channels

- Let $\mathcal{Z}_p$ be a Z-dephasing channel:

$$\mathcal{Z}_p(\rho) = (1-p)\rho + pZ\rho Z.$$

- $\mathcal{Z}_p$ is **degradable** for all p, and $Q(\mathcal{Z}_p) = Q^{(1)}(\mathcal{Z}_p) = 1 - h(p)$.

- Let $\mathcal{E}_q$ be an erasure channel:

$$\mathcal{E}_q(\rho) = (1-q)\rho + q\,\mathrm{Tr}(\rho)|e\rangle\langle e|.$$

- The erasure flag $|e\rangle$ is orthogonal to the input space:

$$\langle e|\rho|e\rangle = 0 \text{ for all } \rho.$$

- $\mathcal{E}_q$ is **degradable** for $q \in [0, 1/2]$ and **antideg.** for $q \in [1/2, 1]$.

Introducing: the dephrasure channel

- Define the **dephrasure channel** $\mathcal{N}_{p,q} := \mathcal{E}_q \circ \mathcal{Z}_p$,

$$\mathcal{N}_{p,q}(\rho) = (1-q)\underbrace{\big((1-p)\rho + pZ\rho Z\big)}_{\text{deg.}} + q\,\underbrace{\mathrm{Tr}(\rho)|e\rangle\langle e|}_{\text{antideg.}}.$$

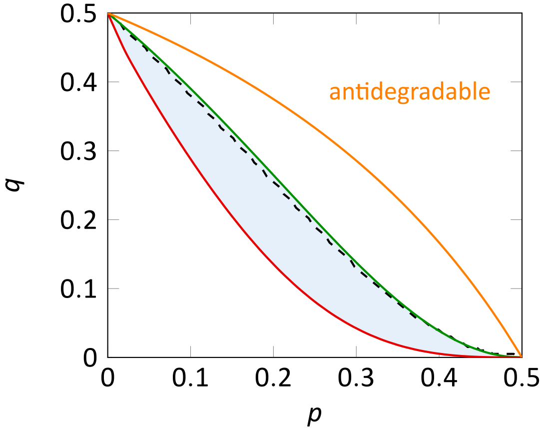
- This is a flagged channel, since $\langle e|\rho|e\rangle = 0$ for all ρ.

- Restrict to $p, q \in [0, 1/2]$ from now on.

- $Q^{(1)}(\mathcal{N}_{p,q}) = \max_{|\varphi\rangle_{AA'}} \left\{(1-q)I(A\rangle B)_{\mathcal{Z}_p(\varphi)} - qS(\varphi_A)\right\}$.

Channel coherent information

- $Q^{(1)}(\mathcal{N}_{p,q}) = \max_{|\varphi\rangle_{AA'}} \left\{(1-q)I(A\rangle B)_{\mathcal{Z}_p(\varphi)} - qS(\varphi_A)\right\}.$



- - - $Q^{(1)}(\mathcal{N}_{p,q}) = 0$

inside green line:
optimizing state φ_A
diagonal in Z-basis

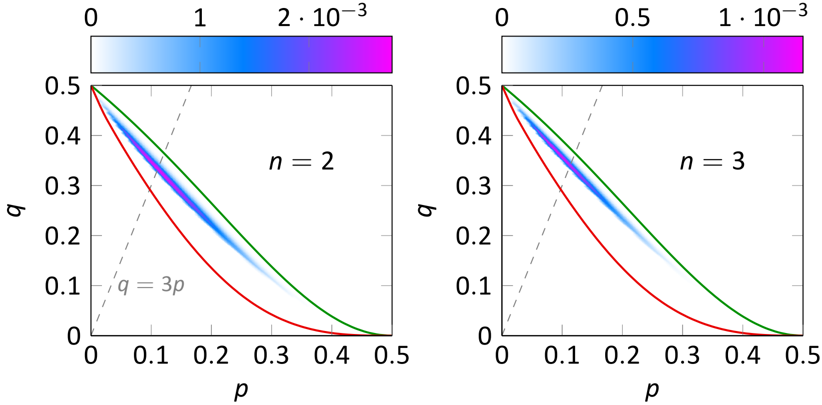
inside red line:
completely mixed state
maximizes $Q^{(1)}(\mathcal{N}_{p,q})$

within blue region:
superadditivity!

Superadditivity of coherent information

- Weighted repetition code $|\varphi_n\rangle := \sqrt{\lambda}|0\rangle^{\otimes n+1} + \sqrt{1-\lambda}|1\rangle^{\otimes n+1}$.
- Plot for $n = 2, 3$ of the non-negative part of

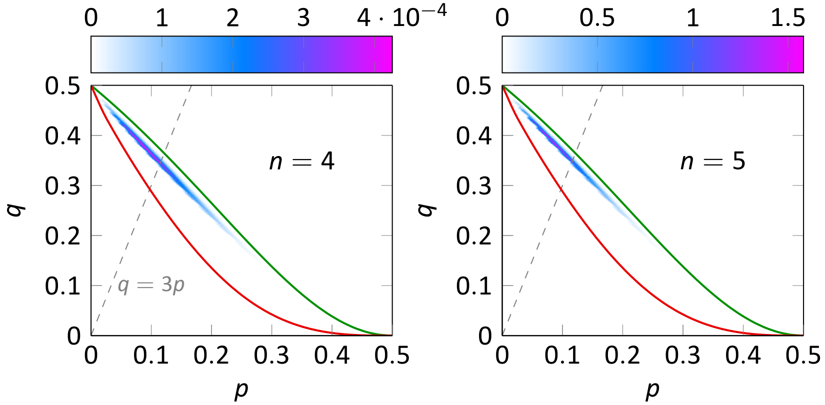
$$\frac{1}{n} \max_{\lambda} I(A\rangle B^n)_{\mathcal{N}_{p,q}^{\otimes n}(\varphi_n)} - Q^{(1)}(\mathcal{N}_{p,q})$$



Superadditivity of coherent information

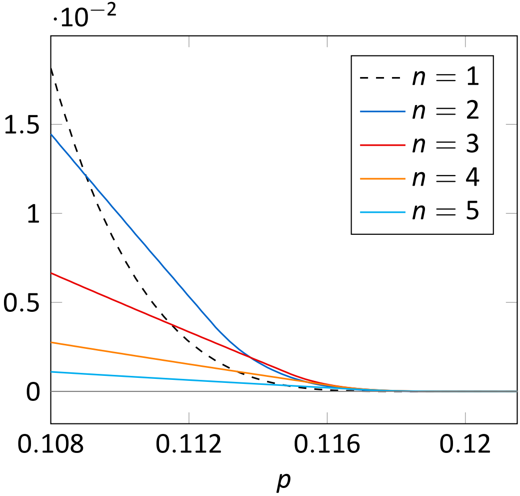
- Weighted repetition code $|\varphi_n\rangle := \sqrt{\lambda}|0\rangle^{\otimes n+1} + \sqrt{1-\lambda}|1\rangle^{\otimes n+1}$.
- Plot for $n = 4, 5$ of the non-negative part of

$$\frac{1}{n}\max_{\lambda} I(A\rangle B^n)_{\mathcal{N}_{p,q}^{\otimes n}(\varphi_n)} - Q^{(1)}(\mathcal{N}_{p,q})$$



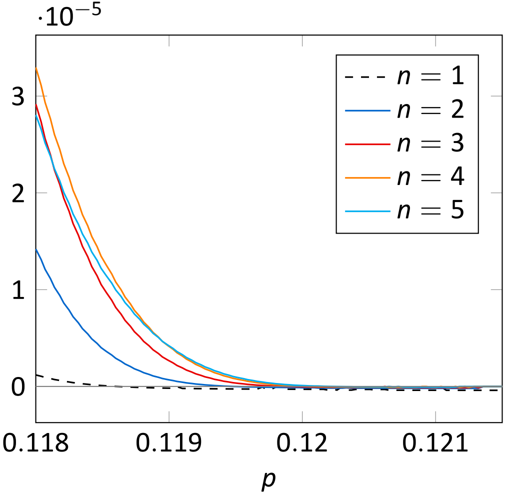
Superadditivity of coherent information

- Plot for $n \in \{1, \ldots, 5\}$ of $\frac{1}{n} \max_{\lambda} I(A\rangle B^n)_{\mathcal{N}_{p,3p}^{\otimes n}(\varphi_n)}$



Superadditivity of coherent information

- Plot for $n \in \{1, \ldots, 5\}$ of $\frac{1}{n} \max_{\lambda} I(A\rangle B^n)_{\mathcal{N}_{p,\,3p}^{\otimes n}(\varphi_n)}$



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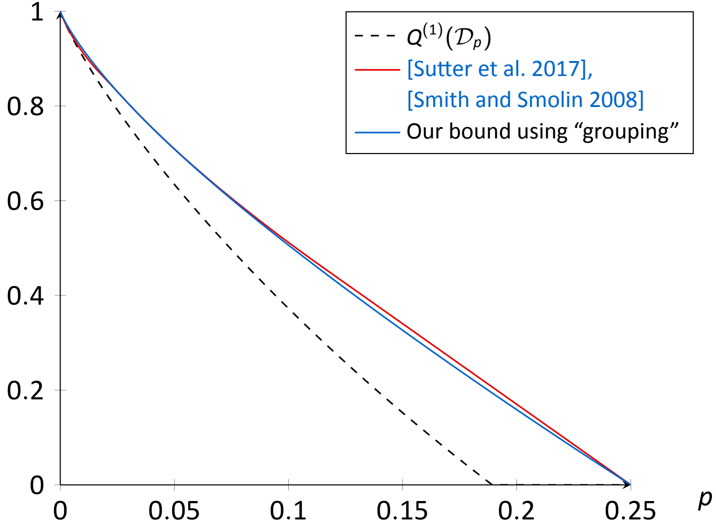
Summary

- Distillable entanglement and quantum capacity are given by regularized formulae and **intractable to compute**.
- **Single-letter formulae** for **degradable** and **antidegradable** states/channels.
- In both settings, our bound is given in terms of decomposition of a state into degradable and antidegradable parts.
- Bound is optimal in a certain way, as demonstrated with the **dephrasure channel**.
- Dephrasure channel exhibits **superadditivity of coherent information** for two uses.

Upper and lower bounds on $Q(\mathcal{D}_p)$



Open questions

- Our bound on the quantum capacity of depolarizing channel is **not tight** - can we improve the upper bound?
- Alternatively, can we find **better quantum codes** with higher superadditivity of coherent information?
  (ongoing work with Johannes Bausch, Graeme Smith)
- $Q(\mathcal{D}_p) = 0$ for $p \geq 1/4$ due to antidegradability/no-cloning, but where is the **true zero point** for $Q(\mathcal{D}_p)$?
- Same for dephrasure channel (we have numerical evidence for codes outperforming the weighted repetition code).
- Dephrasure channel is a toy model $\rightarrow$ **physical application**?

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Thank you very much for your attention!