A limit of the quantum Rényi divergence

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Motivation

Quantum relative entropy $D(\rho\|\sigma) = \mathrm{Tr}(\rho(\log\rho - \log\sigma))$

- significance in asymptotic hypothesis testing
  $\longrightarrow$ Quantum Stein's Lemma
- parent quantity for entropic quantities such as conditional entropy and mutual information

Generalization: α-relative Rényi entropy

$$D_\alpha(\rho\|\sigma) = \frac{1}{\alpha - 1}\log\mathrm{Tr}(\rho^\alpha\sigma^{1-\alpha})$$

- satisfies $\lim_{\alpha\to 1} D_\alpha(\rho\|\sigma) = D(\rho\|\sigma)$

Useful relative entropies in one-shot QIT:

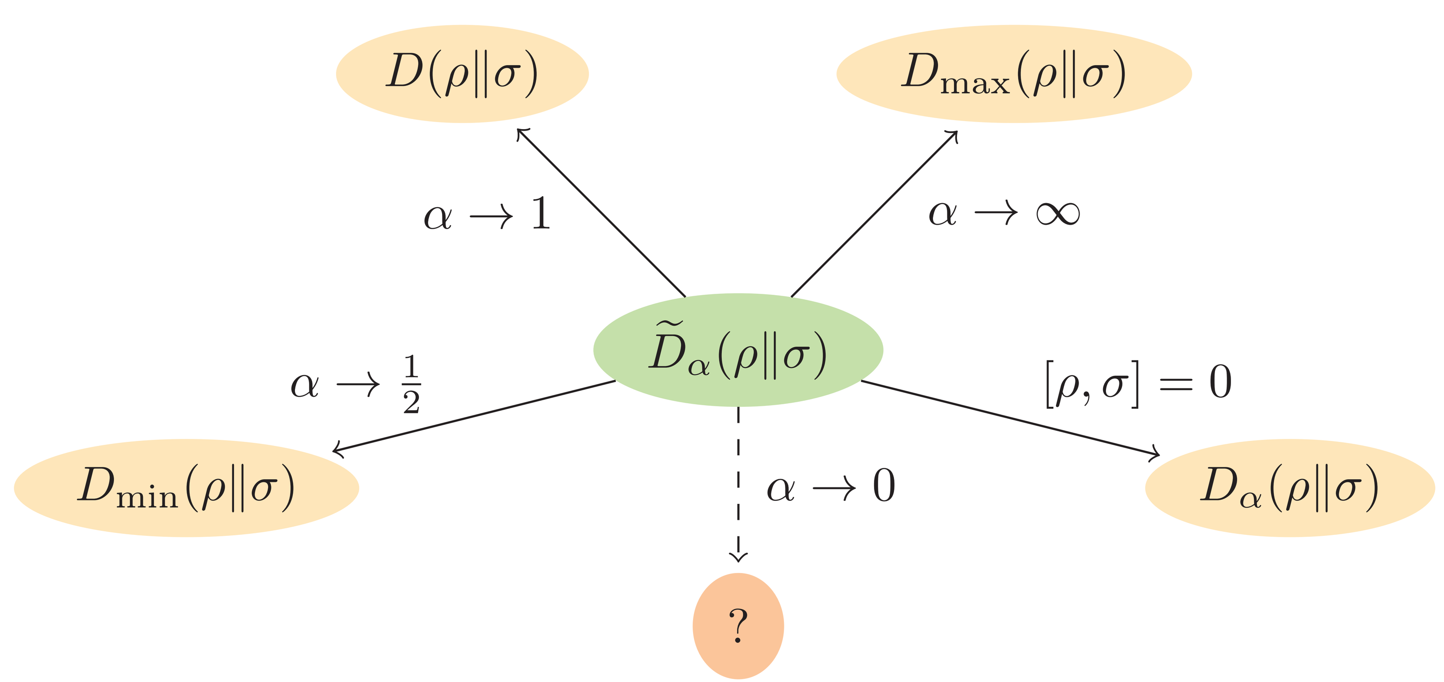
- min-entropy $D_{\min}(\rho\|\sigma) = -2\log\|\sqrt{\rho}\sqrt{\sigma}\|_1$
- max-entropy $D_{\max}(\rho\|\sigma) = \inf\{\gamma : \rho \leq 2^\gamma\sigma\}$
  - quantifies optimal average success probability in multiple state discrimination [3]
- 0-entropy $D_0(\rho\|\sigma) = -\log\mathrm{Tr}(\Pi_\rho\sigma)$
  - entanglement cost of *perfect* one-shot entanglement dilution is given in terms of $D_0(\rho\|\sigma)$ [2]
  - arises in quantum hypothesis testing as $2^{-D_0(\rho\|\sigma)} = \mathrm{Prob}(\text{type II error}|\text{type I error} = 0)$

Quantum Rényi divergence

Definition. *For a state ρ and $\sigma \geq 0$ with $\mathrm{supp}\,\rho \subseteq \mathrm{supp}\,\sigma$,*

$$\widetilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha-1}\log\mathrm{Tr}\left(\sigma^{\frac{1-\alpha}{2\alpha}}\rho\sigma^{\frac{1-\alpha}{2\alpha}}\right)^\alpha$$

"Super parent" for above entropies:



What about the limit $\alpha \to 0$?

Naïve guess: $\lim_{\alpha\to 0}\widetilde{D}_\alpha(\rho\|\sigma) = D_0(\rho\|\sigma)$

- true for $[\rho,\sigma] = 0$, since then $\widetilde{D}_\alpha(\rho\|\sigma) = D_\alpha(\rho\|\sigma)$
- also true under a condition on supports, see Result A [4]
- Result B [1]: explicit expression for $\lim_{\alpha\to 0}\widetilde{D}_\alpha(\rho\|\sigma)$

References

[1] K.M.R. Audenaert, N. Datta, *α-z-relative Rényi entropies*, arXiv:1310.7178 [quant-ph] (2013)

[2] F. Buscemi, N. Datta, *Entanglement cost in practical scenarios*, Phys. Rev. Lett. **106**, 130503 (2011)

[3] N. Datta, *Min- and Max-Relative Entropies and a New Entanglement Monotone*, IEEE Trans. on Inf. Th. **55** (2009), 2816-2826

[4] N. Datta, F. Leditzky, *A limit of the quantum Rényi divergence*, J. Phys. A: Math. Theor. **47**, 045304 (2014)

Result A

If $\mathrm{supp}\,\rho = \mathrm{supp}\,\sigma$, then

$$\lim_{\alpha\to 0}\widetilde{D}_\alpha(\rho\|\sigma) = D_0(\rho\|\sigma) = -\log\mathrm{Tr}(\Pi_\rho\sigma).$$

Proof of Result A

Upper bound:

- $\widetilde{D}_\alpha(\rho\|\sigma) \leq D_\alpha(\rho\|\sigma)$ for all $\alpha > 0$ and $\mathrm{supp}\,\rho \subseteq \mathrm{supp}\,\sigma$ by virtue of the Araki-Lieb-Thirring inequality:

$$\mathrm{Tr}(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{rq} \leq \mathrm{Tr}(B^{\frac{r}{2}}A^rB^{\frac{r}{2}})^q \quad \text{for } q \geq 0, r \geq 1$$
$$\mathrm{Tr}(B^{\frac{r}{2}}AB^{\frac{r}{2}})^q \geq \mathrm{Tr}(B^{\frac{1}{2}}A^rB^{\frac{1}{2}})^{rq} \quad \text{for } q \geq 0, 0 \leq r \leq 1$$

Lower bound:

- employ pinching technique: $\mathcal{E}_\sigma : \rho \mapsto \sum_{i=1}^n |\psi_i\rangle\langle\psi_i|\rho|\psi_i\rangle\langle\psi_i|$ where $\sigma = \sum_{i=1}^n s_i|\psi_i\rangle\langle\psi_i|$ is the spectral decomposition of σ
- pinching lemma: $\rho \leq n\mathcal{E}_\sigma(\rho)$
- $\lim_{\alpha\to 0}\widetilde{D}_\alpha(\rho\|\sigma) \geq D_0(\rho\|\sigma)$ *only if* $\mathrm{supp}\,\rho = \mathrm{supp}\,\sigma$

Counter-example

Consider the states $\rho = |0\rangle\langle 0|$ and $\sigma = |+\rangle\langle +|$, then $D_0(\rho\|\sigma) = 1$ whereas $\widetilde{D}_\alpha(\rho\|\sigma) \xrightarrow{\alpha\to 0} 0$.

Result B

Let ρ be a state and $\sigma \geq 0$ with $\mathrm{supp}\,\rho \subseteq \mathrm{supp}\,\sigma$. Further, let $\sigma = \sum_{i=1}^n \lambda_i|i\rangle\langle i|$ be the eigenvalue decomposition of σ. Then,

$$\lim_{\alpha\to 0}\widetilde{D}_\alpha(\rho\|\sigma) = -\log\left(\sum_{j=1}^s \lambda_{i_j}\right)$$

where $(i_1,\dots,i_s) = \arg\max\left\{\sum_{j=1}^s \lambda_{i_j} : \{\Pi_\rho|i_j\rangle\} \text{ is lin. indep.}\right\}$ and $s = \mathrm{rk}(\Pi_\rho\sigma)$.

Proof of Result B

The proof consists of the following key steps ($\lim \equiv \lim_{\alpha\to 0}$):

- The bound $a\Pi_\rho \leq \rho \leq \Pi_\rho$, where a is the smallest non-zero eigenvalue of ρ, shows that the limit only depends on σ and Π_ρ.
- $\lim\widetilde{D}_\alpha(\rho\|\sigma) = -\log\mathrm{Tr}(\lim Z_\alpha)$ with $Z_\alpha := (\Pi_\rho\sigma^{\frac{1}{\alpha}}\Pi_\rho)^\alpha$
- Let $\sigma = \sum_i \lambda_i|i\rangle\langle i|$ be the spectral decomposition of σ where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, and define the vectors $|u_i\rangle := \Pi_\rho|i\rangle$.
- The largest eigenvalue μ_1 of $\lim Z_\alpha$ is given by $\max_j \lambda_j$ such that $|u_j\rangle \neq 0$.
- Find the product of the two largest eigenvalues of $\lim Z_\alpha$
  $\longrightarrow$ compute the largest eigenvalue $\mu_2 = \lambda_{i_1}\lambda_{i_2}$ of $\lim Z_\alpha^{\wedge 2}$ for which $|u_{i_1}\rangle \wedge |u_{i_2}\rangle \neq 0$ (equivalently, $|u_{i_1}\rangle$ and $|u_{i_2}\rangle$ are linearly independent (LI)).
- Continue this process to obtain μ_k for $k \leq s$, where μ_k is the largest eigenvalue of $\lim Z_\alpha^{\wedge k}$ such that $|u_{i_1}\rangle \wedge \cdots \wedge |u_{i_k}\rangle \neq 0$.
- μ_k is the product of the k largest eigenvalues of $\lim Z_\alpha$, hence

$$\mathrm{Tr}(\lim Z_\alpha) = \mu_1 + \frac{\mu_2}{\mu_1} + \frac{\mu_3}{\mu_2} + \cdots + \frac{\mu_s}{\mu_{s-1}}$$
$$= \max_{i_1,\dots,i_s}\left\{\sum_j \lambda_{i_j} \mid \{|u_{i_j}\rangle\} \text{ is LI}\right\}.$$