Degradable states and one-way entanglement distillation

based on arXiv:1701.03081

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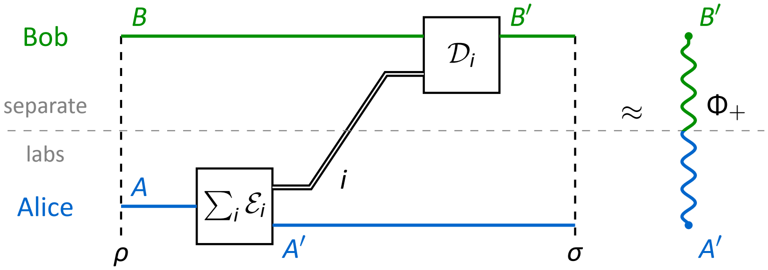
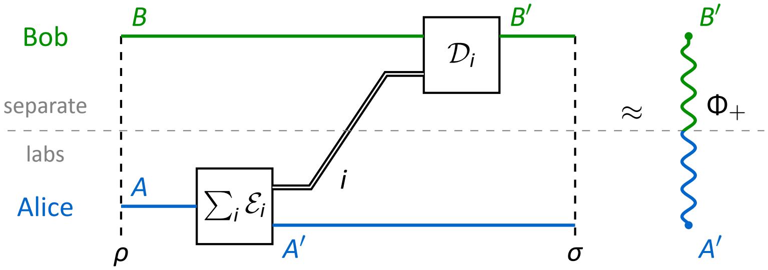
Entanglement distillation using 1-LOCC



- Alice and Bob share n copies of a bipartite (mixed) state ρ_{AB}.
- **Goal:** Distill m_n copies of an MES $|\Phi_+\rangle \sim |00\rangle + |11\rangle$.
- Alice applies an **instrument** $\sum_i \mathcal{E}_i$ to A, consisting of a **quantum output** A' and a **classical output** i that she sends to Bob.
- Conditional on message i, Bob applies some **decoding** $\mathcal{D}_i$ to B.

Entanglement distillation using 1-LOCC



- **Input state:** $\rho_{AB}^{\otimes n} \xrightarrow{\sum_i \mathcal{E}_i \otimes \mathcal{D}_i} \sigma_{A'B'}^n$, **target state:** $|\Phi_+\rangle^{\otimes m_n}$.
- Rate $\lim_{n\to\infty} \frac{m_n}{n}$ is **achievable**, if $\left\|\sigma_{A'B'}^n - \Phi_+^{\otimes m_n}\right\|_1 \xrightarrow{n\to\infty} 0$.
- **One-way distillable entanglement**:

$$D_{\to}(\rho_{AB}) = \sup\{R\colon\ R \text{ is achievable}\}.$$

Entanglement distillation using 1-LOCC

- **Hashing bound** [Devetak and Winter 2005]:

$$D_{\to}(\rho_{AB}) \geq I(A\rangle B)_\rho,$$

where $I(A\rangle B)_\rho = S(B)_\rho - S(AB)_\rho$ is the coherent information.

- Define the following **single-letter quantity**:

$$D^{(1)}_{\to}(\rho_{AB}) := \sup_{\substack{\mathcal{E}:\, A\to A'M \\ \text{instr.}}} I(A'\rangle BM)_{(\mathcal{E}\otimes\mathrm{id})(\rho)}.$$

- **Coding theorem** [Devetak and Winter 2005]:

$$D_{\to}(\rho_{AB}) = \lim_{n\to\infty} \frac{1}{n} D^{(1)}_{\to}(\rho_{AB}^{\otimes n})$$

- Regularization renders **computation** of distillable entanglement **infeasible** in most cases!

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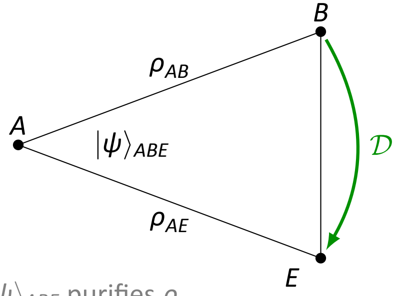
Useful and useless states for entanglement distillation

- **Hashing bound**: $D_{\rightarrow}(\rho_{AB}) \geq I(A\rangle B)$.
- For which states ρ_{AB} is this optimal?

  $\longrightarrow$ **degradable states** [Devetak and Shor 2005; Smith et al. 2008]



$|\psi\rangle_{ABE}$ purifies ρ_{AB}

degradable:

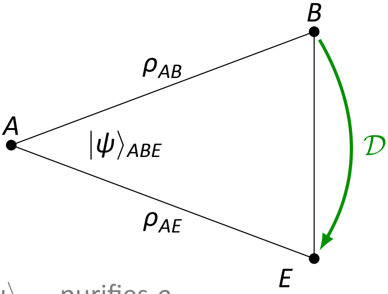
$\exists \mathcal{D}: B \rightarrow E$ s.t.

$\rho_{AE} = (\mathrm{id}_A \otimes \mathcal{D})(\rho_{AB})$

Useful and useless states for entanglement distillation

- Degradable states: $D^{(1)}_{\rightarrow}(\rho_{AB}) = \sup_{\mathcal{E}} I(A'\rangle BM)_{\mathcal{E}(\rho)} = I(A\rangle B)_{\rho}$
- Coherent information is additive: $D^{(1)}_{\rightarrow}(\rho_{AB}^{\otimes n}) = n\, I(A\rangle B)_{\rho}$.
- **Single-letter distillable entanglement:**

$$D_{\rightarrow}(\rho_{AB}) = \lim_{n\rightarrow\infty} \frac{1}{n} D^{(1)}_{\rightarrow}(\rho_{AB}^{\otimes n}) = I(A\rangle B)_{\rho}.$$



$|\psi\rangle_{ABE}$ purifies ρ_{AB}

degradable:

$\exists \mathcal{D}: B \rightarrow E$ s.t.

$\rho_{AE} = (\mathrm{id}_A \otimes \mathcal{D})(\rho_{AB})$

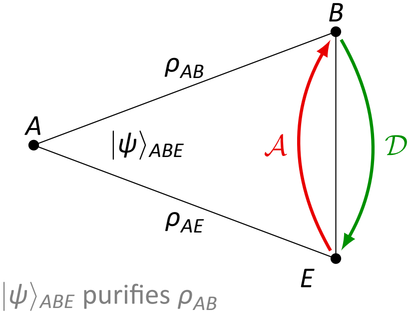
Useful and useless states for entanglement distillation

- On the other hand, which states are useless?

  $\longrightarrow$ **antidegradable states**

- These states always have $I(A\rangle B)_\rho \leq 0$ and $D^{(1)}_{\rightarrow}(\rho_{AB}) \leq 0$.

- **Undistillable:** $D_{\rightarrow}(\rho_{AB}) = 0$.



degradable:

$\exists \mathcal{D}: B \rightarrow E$ s.t.

$\rho_{AE} = (\mathrm{id}_A \otimes \mathcal{D})(\rho_{AB})$

antidegradable:

$\exists \mathcal{A}: E \rightarrow B$ s.t.

$\rho_{AB} = (\mathrm{id}_A \otimes \mathcal{A})(\rho_{AE})$

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Bounding the distillable entanglement

- **Crucial observation** [Wolf and Pérez-García 2007]:
  Regularized quantities such as $D_{\rightarrow}(\cdot)$ are **convex on mixtures** of states with **additive** $D_{\rightarrow}(\cdot)$.

Main result

Let $\rho_{AB} = \sum_i p_i \omega_i + \sum_i q_i \tau_i$, where the ω_i are **degradable** and the τ_i are **antidegradable**. Then,

$$D_{\rightarrow}(\rho_{AB}) \leq \sum_i p_i \, I(A\rangle B)_{\omega_i}.$$

- Such decompositions always exist:
  Every pure state is (trivially) degradable.
- Pure-state decompositions: $\min \sum_i p_i \, I(A\rangle B)_{\omega_i} = E_F(\rho_{AB})$.

Finding good decompositions

- **Challenge**: Find good decompositions into (mixed) degradable and antidegradable parts.
- Easy for **2-qubit states**:
  Every 2-qubit state of rank 2 is either degradable or antidegradable. [Wolf and Pérez-García 2007]
- **Recipe:** $(\psi_i \equiv |\psi_i\rangle\langle\psi_i|)$
  - $\rho_{AB} = p_1\,\psi_1 + p_2\,\psi_2 + p_3\,\psi_3 + p_4\,\psi_4$
  - $\omega_1 := \frac{p_1}{p_1+p_2}\psi_1 + \frac{p_2}{p_1+p_2}\psi_2$ and $\omega_2 := \frac{p_3}{p_3+p_4}\psi_3 + \frac{p_4}{p_3+p_4}\psi_4$
  - Then, rank $\omega_i = 2$ and $\rho_{AB} = (p_1+p_2)\omega_1 + (p_3+p_4)\omega_2$.
  - Apply our main result to this decomposition.

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Depolarizing channel

- Simple, but important qubit error model:

$$\mathcal{D}_p(\rho) = (1-p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z),$$

where $p \in [0, 1]$ and X, Y, Z are the Pauli operators.

- Despite its simplicity, **quantum capacity** $Q(\mathcal{D}_p)$ is **unknown**.

($Q(\mathcal{N}) :=$ max. rate at which entanglement can be generated through $\mathcal{N}$)

- What we do know: $\mathcal{D}_p$ is **teleportation-simulable**,

entanglement generation through $\mathcal{D}_p$

$\Longleftrightarrow$ [Bennett et al. 1996]

one-way entanglement distillation from Choi state $\mathcal{J}(\mathcal{D}_p)$

- In other words: $Q(\mathcal{D}_p) = D_{\rightarrow}(\mathcal{J}(\mathcal{D}_p))$

Bounding quantum capacity of depolarizing channel

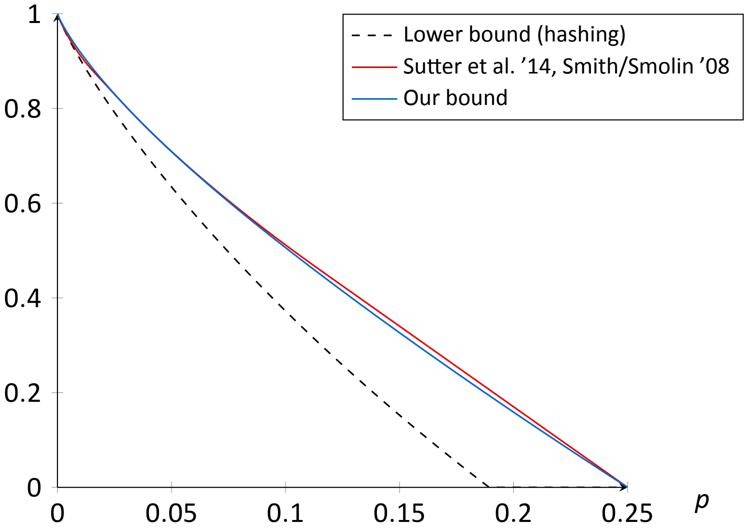
- Choi state $\mathcal{J}(\mathcal{D}_p)$ of depolarizing channel is an **isotropic state** with $U \otimes \bar{U}$ symmetry.
- Our bound can be phrased as a **convex roof extension**. Symmetry simplifies computation! [Vollbrecht and Werner 2001]

Application: Upper bound on $Q(\mathcal{D}_p)$ for $p \in (0, 1/4)$

$$Q(\mathcal{D}_p) = D_{\rightarrow}(\mathcal{J}(\mathcal{D}_p))$$
$$\leq \min\{I(A\rangle B)_\rho : \rho_{AB} \text{ degradable}, \langle \Phi_+ | \rho_{AB} | \Phi_+ \rangle = 1 - p\}$$

- **Bad news:** Non-convex optimization problem, since **set of degradable states is not convex**.
- **Good news:** In low dimensions ($d = 2, 3$) we can still solve it numerically.

Bounding quantum capacity of depolarizing channel



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Conclusion

- One-way distillable entanglement $D_{\rightarrow}(\cdot)$ hard to compute in most cases.
- Main result: upper bound in terms of decompositions of a state into degradable and antidegradable states.
- Easy to compute in low dimensions and for states with symmetries.
- Application to depolarizing channel: strong upper bound on quantum capacity in high-noise regime.
- In arXiv:1701.03081 we also develop a similar framework for two-way entanglement distillation.

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Thank you very much for your attention!