

On the quantum capacity of the qubit depolarizing channel

based on [arXiv:1705.04335](https://arxiv.org/abs/1705.04335) and [arXiv:1701.03081](https://arxiv.org/abs/1701.03081)

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Qubit depolarizing channel

- ▶ Assume that we have a qubit in a state ρ .
- ▶ Fixing some basis $\{|0\rangle, |1\rangle\}$, we consider the following errors:

error	action	operator
bit flip	$ 0\rangle \mapsto 1\rangle$ $ 1\rangle \mapsto 0\rangle$	$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
phase flip	$ 0\rangle \mapsto 0\rangle$ $ 1\rangle \mapsto - 1\rangle$	$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
combined flip	$ 0\rangle \mapsto i 1\rangle$ $ 1\rangle \mapsto -i 0\rangle$	$Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Qubit depolarizing channel

- ▶ **Depolarizing channel** \mathcal{D}_p : each of the three errors (bit flip, phase flip, combined flip) occurs with the same probability $p/3$:

$$\mathcal{D}_p(\rho) := (1 - p)\rho + \frac{p}{3}(X\rho X + Y\rho Y + Z\rho Z).$$

- ▶ Alternatively: \mathcal{D}_p replaces the input with the completely mixed state $\pi_2 := \frac{1}{2}I_2$ with "probability" $q = 4p/3$:

$$\mathcal{D}_p(\rho) = (1 - q)\rho + q\pi_2.$$

- ▶ Exact expression for classical capacity of \mathcal{D}_p is known. [King 2003]
- ▶ Exact expression for **quantum capacity of \mathcal{D}_p is unknown.**

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Entanglement generation

- ▶ Alice and Bob are connected by a quantum channel $\mathcal{N}: A \rightarrow B$ that they can use n times.
- ▶ **Goal:** Generate entanglement in the form of m_n ebits $|\Phi\rangle \sim |00\rangle + |11\rangle$.
- ▶ **Protocol:** Alice prepares a pure state in her lab, sends one half through n copies of \mathcal{N} , Bob applies some decoding.
- ▶ **Quantum capacity** $Q(\mathcal{N})$: largest possible rate at which ebits can be generated with vanishing error.

Quantum capacity

- ▶ **LSD formula** [Lloyd 1997; Shor 2002; Devetak 2005]:

$$Q(\mathcal{N}) = \lim_{n \rightarrow \infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n}), \quad (*)$$

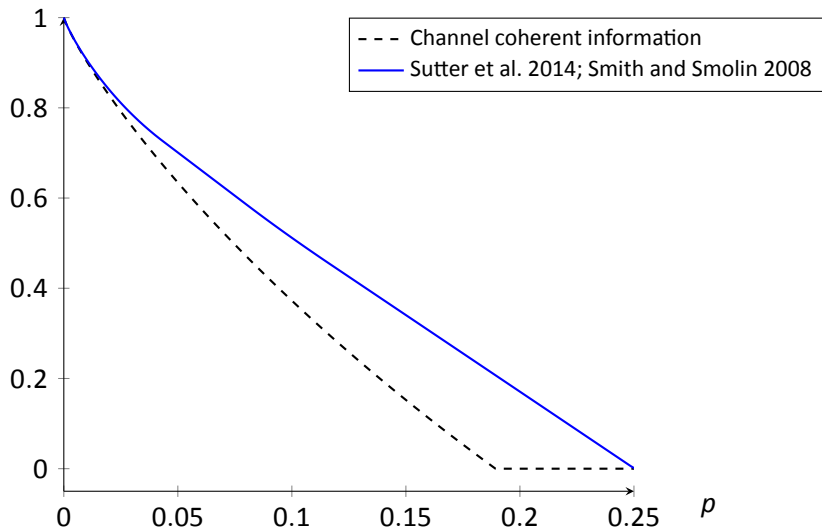
where **channel coherent information** $Q^{(1)}(\mathcal{N})$ is defined as

$$Q^{(1)}(\mathcal{N}) := \max_{|\psi\rangle_{A'A}} I(A'B)_{(\text{id} \otimes \mathcal{N})(\psi)},$$

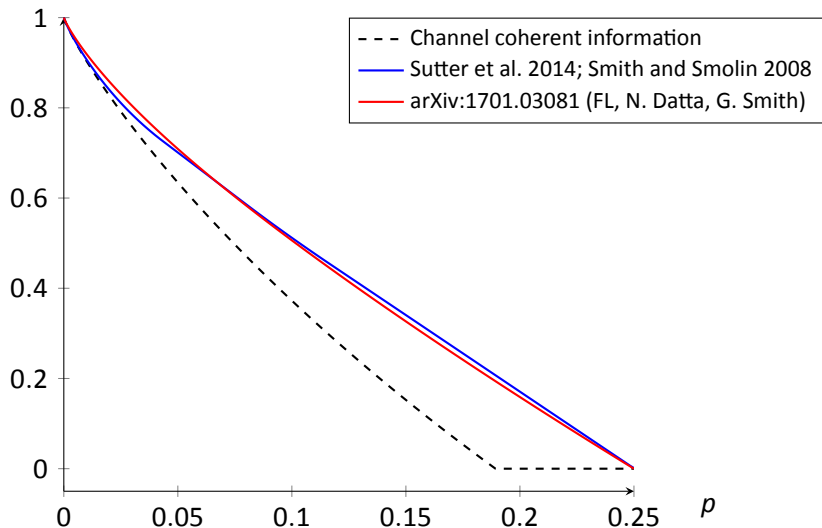
with the coherent information $I(A'B)_\rho = S(B)_\rho - S(AB)_\rho$.

- ▶ **Hashing bound:** $Q(\mathcal{N}) \geq Q^{(1)}(\mathcal{N})$ [Devetak and Winter 2005]
- ▶ **Regularized formula** (*) is in general **intractable to compute**, in particular for \mathcal{D}_ρ .
- ▶ Hence, we are interested in lower and upper bounds on $Q(\mathcal{D}_\rho)$.

Upper and lower bounds on $Q(\mathcal{D}_\rho)$



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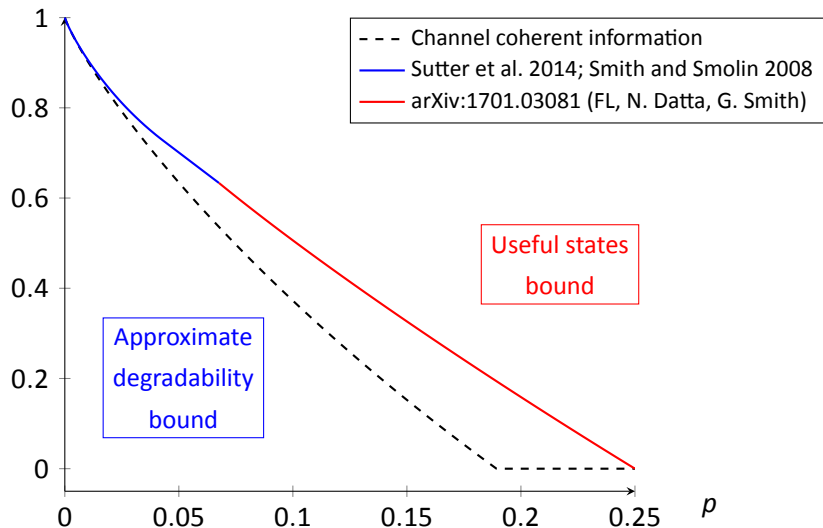


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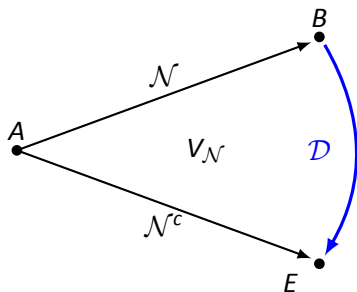
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Low depolarizing noise

- ▶ For low depolarizing noise, i.e., p close to 0, the depolarizing channel is "almost" the identity channel:

$$\mathcal{D}_p = (1 - p) \text{id} + \frac{p}{3}(\text{Pauli errors})$$

- ▶ Capacity of the identity channel is known: $Q(\text{id}) = \log d$.
- ▶ Particularly simple example of a **degradable channel**:



degradable:

$$\exists \mathcal{D}: B \rightarrow E \text{ s.t.}$$

$$\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$$

$V_{\mathcal{N}}: A \rightarrow BE$ is an isometry s.t.

$$\mathcal{N} = \text{Tr}_E(V \cdot V^\dagger),$$

$$\mathcal{N}^c = \text{Tr}_B(V \cdot V^\dagger).$$

Degradability and quantum capacity

- ▶ Recall: Channel coherent information

$$Q^{(1)}(\mathcal{N}) := \max_{|\psi\rangle_{A'A}} I(A'B)_{(\text{id} \otimes \mathcal{N})(\psi)}.$$

- ▶ Recall: LSD-formula

$$Q(\mathcal{N}) = \lim_{n \rightarrow \infty} \frac{1}{n} Q^{(1)}(\mathcal{N}^{\otimes n}).$$

- ▶ **Degradable channels:** $Q^{(1)}(\cdot)$ is weakly additive,

$$Q^{(1)}(\mathcal{N}^{\otimes n}) = nQ^{(1)}(\mathcal{N}) \implies Q(\mathcal{N}) = Q^{(1)}(\mathcal{N}) \quad (*)$$

[Devetak and Shor 2005]

- ▶ **Idea:** Since \mathcal{D}_p is degradable for $p = 0$, is $(*)$ approximately true for $p \gtrsim 0$?

Continuity of quantum capacity

- ▶ What does it mean for two channels to be close?
- ▶ **Diamond norm:** For a linear map Φ with d -dimensional input,

$$\|\Phi\|_{\diamond} := \sup_X \frac{\|(\text{id}_d \otimes \Phi)(X)\|_1}{\|X\|_1}.$$

- ▶ Operational meaning: $\frac{1}{2}\|\Phi - \Psi\|_{\diamond}$ measures distinguishability of channels Φ and Ψ (similar to trace distance and states).
- ▶ Quantum capacity is continuous with respect to the diamond norm [Leung and Smith 2009]:

If $\|\mathcal{N} - \mathcal{M}\|_{\diamond} \leq \epsilon$, then $|Q(\mathcal{N}) - Q(\mathcal{M})| \leq 8\epsilon \log |B| + 4h(\epsilon)$,

where $h(\epsilon)$ is the binary entropy.

Low depolarizing noise and approximate degradability

- ▶ Since $\|\mathcal{D}_p - \text{id}\|_{\diamond} \leq 2p$, we expect $Q(\mathcal{D}_p) \lesssim 1$ for small p .
- ▶ However, continuity bound is not useful for quantitative bounds.
- ▶ Recall: \mathcal{N} is degradable if there is a \mathcal{D} such that $\mathcal{N}^c = \mathcal{D} \circ \mathcal{N}$.

- ▶ **Approximate degradability** [Sutter et al. 2014]

A channel \mathcal{N} is η -degradable if there is another channel \mathcal{D} s.t.

$$\|\mathcal{N}^c - \mathcal{D} \circ \mathcal{N}\|_{\diamond} \leq \eta.$$

- ▶ **Degradability parameter** $\text{dg}(\mathcal{N}) := \text{optimal such } \eta$
- ▶ $\text{dg}(\mathcal{N})$ is the solution of an SDP (also yields optimal \mathcal{D}).

Approximate degradability and quantum capacity

- ▶ **Hashing bound:** $Q(\mathcal{N}) \geq Q^{(1)}(\mathcal{N})$ [Devetak and Winter 2005]
- ▶ This is optimal for **degradable channels:** [Devetak and Shor 2005]

$$Q^{(1)}(\mathcal{N}^{\otimes n}) = nQ^{(1)}(\mathcal{N}) \implies Q(\mathcal{N}) = Q^{(1)}(\mathcal{N})$$

- ▶ **η -degradable channels:**

$$Q^{(1)}(\mathcal{N}) \leq Q(\mathcal{N}) \leq Q^{(1)}(\mathcal{N}) + f(\eta, |E|),$$

where $f(\eta, |E|) = -\eta \log \eta + O(\eta)$.

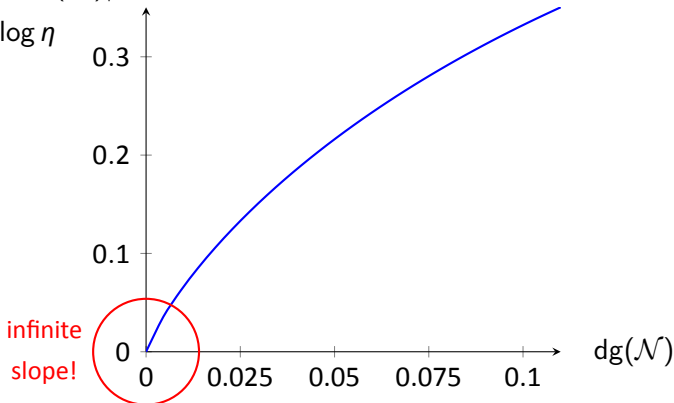
- ▶ Recovers degradable case: $f(\eta, |E|) \xrightarrow{\eta \rightarrow 0} 0$.
- ▶ **Problem:** In the generic case, this is not a very useful approximation.

Approximate degradability and quantum capacity

- **Problem:** In the generic case, this is not a very useful approximation.

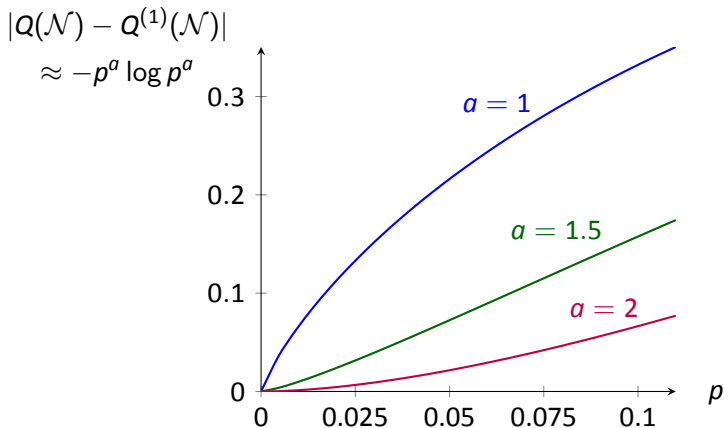
$$|Q(\mathcal{N}) - Q^{(1)}(\mathcal{N})|$$

$$\approx -\eta \log \eta$$



Approximate degradability and quantum capacity

- ▶ **But:** If $\text{dg}(\mathcal{N}) = p^a$ for some underlying parameter p , and $a > 1$
→ approximation is tangent!



Main results

- ▶ **Degradability:** $\text{dg}(\mathcal{D}_\rho) = O(\rho^2)$.

- ▶ **Quantum capacity:**

$$Q(\mathcal{D}_\rho) = 1 - h(\rho) - \rho \log 3 + O(\rho^2 \log \rho^2).$$

- ▶ **Proof idea:** Guess degrading map \mathcal{M}_ρ in $\|\mathcal{D}_\rho^c - \mathcal{M}_\rho \circ \mathcal{D}_\rho\|_\diamond$.
- ▶ **Intuition:** For small ρ , depolarizing channel is almost identity, so complementary channel is almost completely depolarizing.
- ▶ Let's use the **complementary channel** itself as degrading map \mathcal{M}_ρ , and "give a little more to Bob":

$$\mathcal{M}_\rho = \mathcal{D}_{s(\rho)}^c \quad \text{with } s(\rho) = \rho + a\rho^2.$$

- ▶ For $a = 8/3$, this yields $\|\mathcal{D}_\rho^c - \mathcal{D}_{s(\rho)}^c \circ \mathcal{D}_\rho\|_\diamond = O(\rho^2)$.

Usefulness of approximate degradability bound

- ▶ Approximate degradability might be useless in generic case.
- ▶ **However:** Extremely useful for channels with some underlying noise parameter such as \mathcal{D}_p , for which $\text{dg}(\mathcal{D}_p) = O(p^2)$.
- ▶ **Generalizations:**
 - ▶ Same quadratic ansatz works for any generalized Pauli channel \mathcal{N} , proving $\text{dg}(\mathcal{N}) = O(p^2)$.
 - ▶ For any "low-noise channel" \mathcal{N} with $\|\text{id} - \mathcal{N}\|_\diamond \leq \epsilon$, we have

$$\|\mathcal{N}^c - \mathcal{N}^c \circ \mathcal{N}\|_\diamond \leq 2\epsilon^{1.5}.$$

→ Low-noise channels are approximately degraded by their complementary channel!

Approximate degradability of depolarizing channel

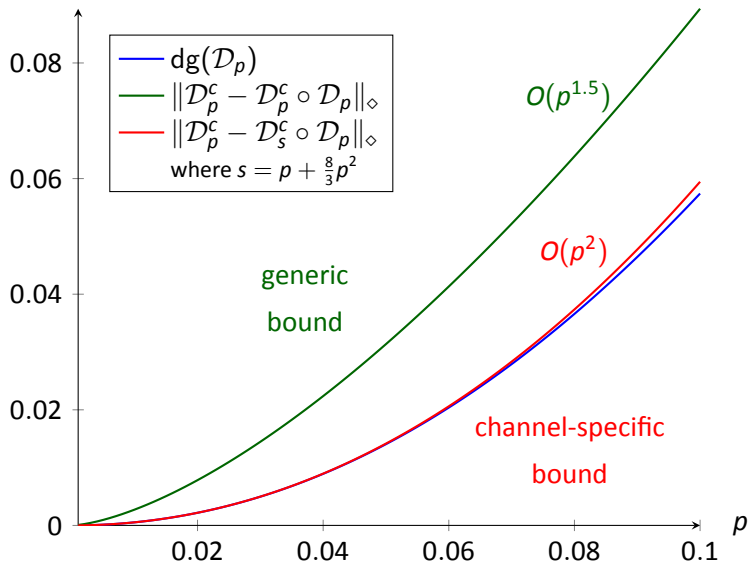
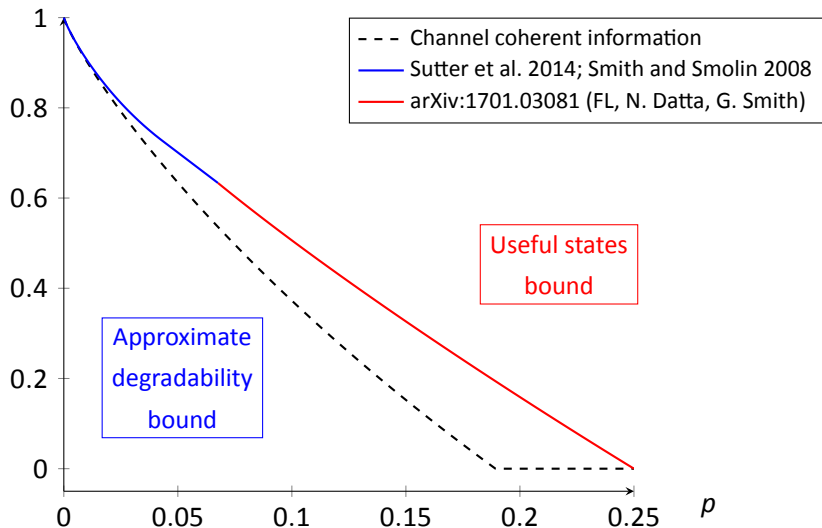


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Upper and lower bounds on $Q(\mathcal{D}_\rho)$



Detour: Entanglement distillation using forward LOCC

- ▶ **Entanglement distillation:** Convert "noisy" entanglement resource ρ_{AB} shared between Alice and Bob into a "clean" form, e.g., ebits $|\Phi\rangle \sim |00\rangle + |11\rangle$.
- ▶ **Protocol:** local operations (LO) and classical communication (CC) between Alice and Bob.
- ▶ Recall that forward CC **does not increase** quantum capacity of a channel [Bennett et al. 1996; Barnum et al. 2000].
- ▶ Restrict to **forward CC** from Alice to Bob **only**.
- ▶ **Distillable entanglement** $D_{\rightarrow}(\rho_{AB})$: largest possible rate at which ebits can be distilled with vanishing error.

Entanglement distillation and entanglement generation

- ▶ **Entanglement distillation** and **entanglement generation** are just the **static** and **dynamic** side of the same coin! [Bennett et al. 1996]
- ▶ **egen** \rightarrow **edist**:
 - ▷ Shared state ρ_{AB} + forward LOCC \rightarrow teleportation scheme [Bennett et al. 1993] \rightarrow noisy channel \mathcal{N}_ρ .
 - ▷ egen code for \mathcal{N}_ρ distills ebits from ρ_{AB} \rightarrow edist protocol.
- ▶ **edist** \rightarrow **egen**:
 - ▷ ebit (in Alice's lab) + channel \mathcal{N} from Alice to Bob \rightarrow shared Choi state $\tau_{\mathcal{N}}$.
 - ▷ edist protocol for $\tau_{\mathcal{N}}$ generates ebits through \mathcal{N} \rightarrow egen code.

Entanglement distillation and entanglement generation

- ▶ These two mappings ($\text{egen} \leftrightarrow \text{edist}$) are inverse to each other for so-called **teleportation-simulable** (TS) channels.
- ▶ **TS channel:** any output state can be obtained using a teleportation protocol run on the Choi state of the channel.
- ▶ Depolarizing channel is TS [Bennett et al. 1996]:

$$Q(\mathcal{D}_\rho) = D_{\rightarrow}(\tau_{\mathcal{D}_\rho})$$

- ▶ From now on, focus on the "static" resource $\tau_{\mathcal{D}_\rho}$ and $D_{\rightarrow}(\tau_{\mathcal{D}_\rho})$.

Useful states for entanglement distillation

- ▶ **Hashing bound** [Devetak and Winter 2005]

$$D_{\rightarrow}(\rho_{AB}) \geq I(A>B)_{\rho}.$$

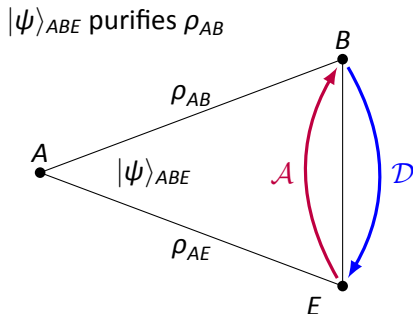
- ▶ **But:** $D_{\rightarrow}(\cdot)$ is also given by a regularized formula [Devetak and Winter 2005], and hence as **intractable to compute** as $Q(\cdot)$.
- ▶ **Solution:** Identify "useful" and "useless" states which make the quantity $D_{\rightarrow}(\cdot)$ behave nicely.
- ▶ **Useful states:** States ρ_{AB} for which hashing is optimal protocol:

$$D_{\rightarrow}(\rho_{AB}) = I(A>B)_{\rho} \geq 0.$$

- ▶ **Useless states:** Undistillable states σ_{AB} , i.e., $D_{\rightarrow}(\sigma_{AB}) = 0$. They always have $I(A>B)_{\sigma} \leq 0$.

Useful states for entanglement distillation

- ▶ The **useful** and **useless** states for one-way entanglement distillation are given by **degradable** and **antidegradable** states.



degradable:

$\exists \mathcal{D}: B \rightarrow E$ s.t.

$$\rho_{AE} = (\text{id}_A \otimes \mathcal{D})(\rho_{AB})$$

antidegradable:

$\exists \mathcal{A}: E \rightarrow B$ s.t.

$$\rho_{AB} = (\text{id}_A \otimes \mathcal{A})(\rho_{AE})$$

Useful states for entanglement distillation

- ▶ **Key observation:** $D_{\rightarrow}(\cdot)$ is convex on degradable and antidegradable states!

(proved by [Wolf and Pérez-García 2007] for quantum capacity)

- ▶ **Main result:** Decompose ρ_{AB} into mixture of degradable (ω_i) and antidegradable (σ_i) states,

$$\rho_{AB} = \sum_i p_i \omega_i + \sum_i p_i \sigma_i, \quad (*)$$

then we have the following upper bound:

$$D_{\rightarrow}(\rho_{AB}) \leq \sum_i p_i I(A \rangle B)_{\omega_i}. \quad (**)$$

- ▶ (*) always exists, since **every pure state is degradable**.
- ▶ Minimizing (**) over all pure-state decompositions of ρ_{AB} yields **entanglement of formation** $E_F(\rho_{AB})$.

Quantum capacity of depolarizing channel

- ▶ **Goal:** Apply this upper bound to Choi state $\tau_{\mathcal{D}_\rho}$ of depolarizing channel to get upper bound on $Q(\mathcal{D}_\rho) = D_{\rightarrow}(\tau_{\mathcal{D}_\rho})$.
- ▶ **Strategy:** Use symmetries of $\tau_{\mathcal{D}_\rho}$!
- ▶ Choi state $\tau_{\mathcal{D}_\rho}$ of depolarizing channel is an **isotropic state**: 1-parameter family of states $\{\text{Iso}(f) : f \in [0, 1]\}$ invariant under all local unitaries of the form $U \otimes U^*$. (here, $f = 1 - \rho$)
- ▶ The symmetry group $\{U \otimes U^* : U \text{ unitary}\}$ defines a bilateral twirl operation

$$\rho_{AB} \longmapsto \int dU (U \otimes U^*) \rho_{AB} (U \otimes U^*)^\dagger = \text{Iso}(f_\rho)$$

where $f_\rho = \langle \Phi | \rho_{AB} | \Phi \rangle$.

Quantum capacity of depolarizing channel

- ▶ By the no-cloning theorem, $Q(\mathcal{D}_p) = 0$ for $p \geq 1/4$.
- ▶ For $p \leq 1/4$, the $U \otimes U^*$ -symmetry of $\tau_{\mathcal{D}_p}$ allows us to restrict to degradable states that "twirl" to $\tau_{\mathcal{D}_p}$:

$$D_{\rightarrow}(\tau_{\mathcal{D}_p}) \leq \inf \{I(A)B\}_\rho : \rho_{AB} \text{ degradable, } \langle \Phi | \rho_{AB} | \Phi \rangle = 1 - p \}.$$

- ▶ **Bad news:** This is a non-convex optimization problem, since the set of degradable states is not convex.
- ▶ **Good news:** We can solve it numerically for low dimensions (qubits, qutrits) \longrightarrow yields advertised upper bound for $d = 2$!

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Conclusion

- ▶ Quantum capacity of depolarizing channel is unknown.
- ▶ Best upper bound for low depolarizing noise: Approximate degradability bound.
- ▶ Approximates desirable properties of degradable channels.
- ▶ Best upper bound for high depolarizing noise: Useful states bound.
- ▶ Based on decomposition of Choi state into degradable and antidegradable parts.

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Thank you very much for your attention!