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4 Further results and outlook
Optimal rates of information-theoretic tasks

- Information-theoretic tasks: source coding, channel coding, 
  
- Operational definition of a rate of the code: compression rate, capacity, 
  
- Coding theorem establishes entropic quantity as optimal rate:

\[
\begin{align*}
\text{(Weak) Converse} & & \text{Achievability} \\
\text{every code has} & & \text{there exists a code} \\
\lim_{n \to \infty} \varepsilon_n > 0 & & \text{with} \lim_{n \to \infty} \varepsilon_n = 0
\end{align*}
\]

- Achievability
  - Coding theorem establishes entropic quantity as optimal rate:

\[
\lim_{n \to \infty} \varepsilon_n = 0
\]

- Strong converse theorems

\[
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\]
Strong converse property

- Weak converse: trade-off between error $\varepsilon_n$ and rate $r$ below optimal rate $r^*$?

- Strong converse theorem: No!
  Every code at rate below the optimal rate fails with certainty!

- Optimal rate satisfies strong converse property.

$$\lim_{n \to \infty} \varepsilon_n$$

\[0 \quad 1\]

\[0 \quad r^* \quad r\]

sharp threshold
**Strong converse property**

**Example: Quantum data compression**

- **Optimal compression rate** [Schumacher 1995]:
  \[ r^* = S(\rho) = - \text{Tr}(\rho \log \rho) \]
- **Strong converse for data compression** [Winter 1999]:
  For every code with rate \( r < S(\rho) \), we have
  \[ \varepsilon_n \geq 1 - \exp(-Kn) \]
  for some \( K > 0 \), and hence \( \lim_{n \to \infty} \varepsilon_n = 1 \) (since \( \varepsilon_n \in [0, 1] \)).

- **Exponential convergence of error**
  \[ \longrightarrow \text{strong converse in the Wolfowitz sense} \ [Wolfowitz 1961]. \]

**How can we prove strong converse theorems?**

- **Winter**: method of types
- **Here**: Rényi entropy approach [Arimoto 1973; Ogawa and Nagaoka 1999]
- **Derive lower bound on error in terms of a Rényi entropic quantity**
Rényi entropies

Definition (Sandwiched Rényi divergence of order $\alpha$)

Let $\alpha \in (0, \infty) \setminus \{1\}$ and $\rho, \sigma$ be quantum states with $\text{supp} \rho \subseteq \text{supp} \sigma$:

$$\tilde{D}_\alpha(\rho\|\sigma) := \frac{1}{\alpha - 1} \log \text{Tr} \left( \sigma^{(1-\alpha)/2\alpha} \rho \sigma^{(1-\alpha)/2\alpha} \right)^{\alpha}. $$

- **Additivity:** $\tilde{D}_\alpha(\rho_1 \otimes \rho_2\|\sigma_1 \otimes \sigma_2) = \tilde{D}_\alpha(\rho_1\|\sigma_1) + \tilde{D}_\alpha(\rho_2\|\sigma_2)$

- **Data processing inequality:** [Beigi 2013; Frank and Lieb 2013]
  For a quantum channel $\Lambda$ and $\alpha \geq 1/2$,
  $$\tilde{D}_\alpha(\rho\|\sigma) \geq \tilde{D}_\alpha(\Lambda(\rho)\|\Lambda(\sigma)).$$

- **Limit property:**
  $$\tilde{D}_\alpha(\rho\|\sigma) \xrightarrow{\alpha \to 1} D(\rho\|\sigma) = \text{Tr}[\rho(\log \rho - \log \sigma)].$$
Rényi entropic quantities

The sandwiched Rényi divergence ($\alpha$-SRD) serves as a parent for the following quantities:

**Definition (Entropic quantities derived from $\alpha$-SRD)**

Let $\alpha > 0$ and $\rho_{AB}$ be a bipartite quantum state with marginal $\rho_A$.

- **Rényi entropy**

  $$S_\alpha(A)_\rho := -\widetilde{D}_\alpha(\rho_A\|1_A)$$

- **Rényi conditional entropy (RCE)**

  $$\tilde{S}_\alpha(A|B)_\rho := -\min_{\sigma_B} \widetilde{D}_\alpha(\rho_{AB}\|1_A \otimes \sigma_B)$$

- **Rényi mutual information (RMI)**

  $$\tilde{I}_\alpha(A;B)_\rho := \min_{\sigma_B} \widetilde{D}_\alpha(\rho_{AB}\|\rho_A \otimes \sigma_B)$$
Rényi entropic quantities

These quantities inherit some of the properties of the $\alpha$-SRD:

- **Additivity** [Hayashi and Tomamichel 2014]

Let $\rho_{A_1B_1}$ and $\sigma_{A_2B_2}$ be quantum states, then

$$
\widetilde{S}_\alpha(A_1A_2|B_1B_2)_{\rho\otimes\sigma} = \widetilde{S}_\alpha(A_1|B_1)_{\rho} + \widetilde{S}_\alpha(A_2|B_2)_{\sigma}
$$

$$
\tilde{I}_\alpha(A_1A_2; B_1B_2)_{\rho\otimes\sigma} = \tilde{I}_\alpha(A_1; B_1)_{\rho} + \tilde{I}_\alpha(A_2; B_2)_{\sigma}.
$$

- **Data processing inequality**

Let $\Lambda: B \rightarrow C$ be a quantum channel, and $\omega_{AC} = (id_A \otimes \Lambda)(\rho_{AB})$, then for $\alpha \geq 1/2$ we have

$$
\widetilde{S}_\alpha(A|B)_{\rho} \leq \widetilde{S}_\alpha(A|C)_{\omega} \quad \quad \quad \quad \quad \tilde{I}_\alpha(A; B)_{\rho} \geq \tilde{I}_\alpha(A; C)_{\omega}.
$$

- **Limit property**

$$
S_\alpha(A)_{\rho} \xrightarrow{\alpha \rightarrow 1} S(A)_{\rho}
$$

$$
\widetilde{S}_\alpha(A|B)_{\rho} \xrightarrow{\alpha \rightarrow 1} S(A|B)_{\rho} = S(AB)_{\rho} - S(B)_{\rho}
$$

$$
\tilde{I}_\alpha(A; B)_{\rho} \xrightarrow{\alpha \rightarrow 1} I(A; B)_{\rho} = S(A)_{\rho} - S(A|B)_{\rho}
$$
We prove the following new properties for these quantities:

**Theorem (Dimension bounds)**

For $\alpha \geq 1/2$ and a tripartite state $\rho_{ABC}$ with $C$ quantum,

\[
\tilde{S}_\alpha(A|BC)_\rho + 2 \log |C| \geq \tilde{S}_\alpha(A|B)_\rho \\
\tilde{I}_\alpha(A; B)_\rho + 2 \log |C| \geq \tilde{I}_\alpha(A; BC)_\rho
\]

whereas for $\rho_{ABX}$ with $X$ classical,

\[
\tilde{S}_\alpha(A|BX)_\rho + \log |X| \geq \tilde{S}_\alpha(A|B)_\rho \\
\tilde{I}_\alpha(A; B)_\rho + \log |X| \geq \tilde{I}_\alpha(A; BX)_\rho.
\]
We prove the following new properties for these quantities:

**Theorem (Fidelity bounds)**

For $\alpha \in (1/2, 1)$, $\beta = \alpha/(2\alpha - 1)$, and bipartite states $\rho_{AB}$ and $\sigma_{AB}$,

$$S_\alpha(A)_\rho - S_\beta(A)_\sigma \geq \frac{2\alpha}{1 - \alpha} \log F(\rho_A, \sigma_A)$$

$$\tilde{S}_\alpha(A|B)_\rho - \tilde{S}_\beta(A|B)_\sigma \geq \frac{2\alpha}{1 - \alpha} \log F(\rho_{AB}, \sigma_{AB})$$

$$\tilde{I}_\beta(A; B)_\rho - \tilde{I}_\alpha(A; B)_\sigma \geq \frac{2\alpha}{1 - \alpha} \log F(\rho_{AB}, \sigma_{AB})$$

where $F(\omega, \tau) := \|\sqrt{\omega} \sqrt{\tau}\|_1$ is the fidelity.

Eq. (*) first appeared in [van Dam and Hayden 2002], and we generalize this result to the Rényi conditional entropy and mutual information.
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State redistribution: protocol

- **target state:** $\psi_{A'B'C'R} \otimes \Phi^m_{T'_A T'_B} \ (A'B'C' \cong ABC)$
- **figure of merit:** $F(\psi \otimes \Phi^m, (D \circ E)(\psi \otimes \Phi^k))$
- **# of qubits sent from Alice to Bob:** $\log |Q|$
- **# of ebits consumed:** $\log |T_A| - \log |T'_A| = \log k - \log m$
  (if $\log k < \log m$, then ebits are gained)
$n$ copies of $\rho_{ABC}$:

- **Initial state:** $\psi^n_{ABCR} \otimes \Phi^{k_n}_{T_A T_B}$

- **Overall map:** Encoding $\mathcal{E}_n$, quantum communication $Q$, and decoding $\mathcal{D}_n$.

- **Target state:** $\psi^n_{A'B'C'R} \otimes \Phi^{m_n}_{T'_A T'_B}$

- **Figure of merit:** $F_n := F(\psi^n \otimes \Phi^{m_n}, (\mathcal{D}_n \circ \mathcal{E}_n)(\psi^n \otimes \Phi^{k_n}))$

- **Quantum communication cost:** $q_n := \frac{1}{n} \log |Q^n|$

- **Entanglement cost:** $e_n := \frac{1}{n}(\log k_n - \log m_n)$
Definition (Achievable rates)

\((e, q)\) is achievable: For \(\rho \equiv \rho_{ABC}\) there is a protocol \(\{(\rho \otimes^n, E_n, D_n)\}_{n \in \mathbb{N}}\) with \(\lim_{n \to \infty} F_n = 1\) and

\[
\limsup_{n \to \infty} e_n = e, \quad \limsup_{n \to \infty} q_n = q.
\]

Theorem (Luo and Devetak 2009; Yard and Devetak 2009)

The pair \((e, q)\) is achievable if and only if

\[
q \geq \frac{1}{2} I(A; R|B)_\rho, \quad q + e \geq S(A|B)_\rho.
\]

Conditional mutual information (CMI): \(I(A; R|B)_\rho = S(A|B)_\rho - S(A|RB)_\rho\)
Main result: strong converse region

\[ q + e \geq S(A|B)_\rho \]

\[ q \geq \frac{1}{2} I(A; R|B)_\rho \]

achievable region

strong conv. reg.

\[ e \]

\[ q \]
State redistribution: Strong converse theorem

Main result:

Theorem

For every state redistribution protocol with initial state $\rho_{ABC}$, we have the following bounds on $F_n$ for all $n \in \mathbb{N}$ and $\alpha \in (1/2, 1)$, setting $\beta = \alpha/(2\alpha - 1)$ and $\kappa(\alpha) = (1 - \alpha)/(2\alpha) > 0$:

$$F_n \leq \exp\left\{ -n\kappa(\alpha) \left[ S_{\beta}(AB)_\rho - S_{\alpha}(B)_\rho - (q_n + e_n) \right] \right\}$$

$$F_n \leq \exp\left\{ -n\kappa(\alpha) \left[ \tilde{S}_{\beta}(R|B)_\rho - \tilde{S}_{\alpha}(R|AB)_\rho - 2q_n \right] \right\}$$

As an alternative to the second bound, we also have

$$F_n \leq \exp\left\{ -n\kappa(\alpha) \left[ \tilde{I}_{\alpha}(R; AB)_\rho - \tilde{I}_{\beta}(R; B)_\rho - 2q_n \right] \right\}$$
State redistribution: Strong converse theorem

Bounds on fidelity ($\beta \equiv \beta(\alpha) = \alpha/(2\alpha - 1)$)

$$F_n \leq \exp \left\{ -n\kappa(\alpha) [S_{\beta}(AB)_{\rho} - S_{\alpha}(B)_{\rho} - (q_n + e_n)] \right\}$$

$$F_n \leq \exp \left\{ -n\kappa(\alpha) \left[ \tilde{S}_{\beta}(R|B)_{\rho} - \tilde{S}_{\alpha}(R|AB)_{\rho} - 2q_n \right] \right\}$$

Optimal rates: $q + e \geq S(A|B)_{\rho}$, $q \geq \frac{1}{2} I(A; R|B)_{\rho}$

- Rényi generalization of conditional entropy:
  $$S_{\beta(\alpha)}(AB)_{\rho} - S_{\alpha}(B)_{\rho} \xrightarrow{\alpha \to 1} S(AB)_{\rho} - S(B)_{\rho} = S(A|B)_{\rho}$$

- Converse region: $q_n + e_n < S(A|B)_{\rho}$

- There is $\alpha_0 \in (1/2, 1)$ such that
  $$C := \kappa(\alpha_0)[S_{\beta(\alpha_0)}(AB)_{\rho} - S_{\alpha_0}(B)_{\rho} - (q_n + e_n)] > 0.$$  

- **Strong converse**: $F_n \leq \exp\{-nC\} \xrightarrow{n \to \infty} 0$
Bounds on fidelity ($\beta \equiv \beta(\alpha) = \alpha/(2\alpha - 1)$)

\[ F_n \leq \exp\left\{-n\kappa(\alpha) \left[ S_{\beta}(AB)_\rho - S_\alpha(B)_\rho - (q_n + e_n) \right] \right\} \]

\[ F_n \leq \exp\left\{-n\kappa(\alpha) \left[ \tilde{S}_{\beta}(R|B)_\rho - \tilde{S}_\alpha(R|AB)_\rho - 2q_n \right] \right\} \]

Optimal rates: $q + e \geq S(A|B)_\rho$, $q \geq \frac{1}{2} I(A; R|B)_\rho$

- Rényi generalization of conditional mutual information:
  \[ \tilde{S}_{\beta(\alpha)}(R|B)_\rho - \tilde{S}_\alpha(R|AB)_\rho \xrightarrow{\alpha \rightarrow 1} S(R|B)_\rho - S(R|AB)_\rho = I(A; R|B)_\rho \]

- Converse region: $2q_n < I(A; R|B)_\rho$

- As before, this yields a strong converse:
  \[ F_n \leq \exp\{-nC\} \xrightarrow{n \rightarrow \infty} 0 \text{ for some } C > 0. \]
Proof sketch

**Strong converse property for** $q_n + e_n$

- **Strategy:** Prove for $n = 1$ and then use additivity of Rényi quantities.
- **Define post-encoding state**

$$|\omega_{C'QBRT_A'T_BE}\rangle := U_{\mathcal{E}} \left(|\psi_{ABCR}\rangle \otimes |\Phi^k_{T_AT_B}\rangle \right)$$

where $U_{\mathcal{E}}$ is a Stinespring isometry of Alice’s encoding map $\mathcal{E} : ACT_A \rightarrow QC'T'_A$ with environment $E$.

- **“Subadditivity” property for Rényi entropies:**

  [van Dam and Hayden 2002] For $\alpha > 0$ and a bipartite state $\rho_{AB}$,

  $$S_\alpha(A)_\rho - \log |B| \leq S_\alpha(AB)_\rho \leq S_\alpha(A)_\rho + \log |B|$$

- For the marginal $\omega_{QBT_B}$, this yields

  $$S_\alpha(QBT_B)_\omega \leq \log |Q| + \log |T_A| + S_\alpha(B)_\rho.$$
Proof idea

- Define final state

\[ |\sigma_{A'B'C'RT_A'T_B'E}⟩ := U_D |\omega C'QBRT_A'T_B'E⟩ \]

where \( U_D \) is a Stinespring isometry of Bob’s decoding map \( D: QBT_B \rightarrow A'B'T_B' \) with environment \( D \).

- Isometric invariance: \( S_{\alpha}(QBT_B)\omega = S_{\alpha}(A'B'T_B'D)\sigma \).

- Uhlmann:

\[ F = F(\psi \otimes \Phi^m, (D \circ E)(\psi \otimes \Phi^k)) \leq F(\sigma_{A'B'T_B'D}, \pi^{m}_{T_A} \otimes \rho_{A'B'} \otimes \chi_D) \]

- Fidelity bound applied to \( \sigma_{A'B'T_B'D} \) and \( \pi^{m}_{T_A} \otimes \rho_{A'B'} \otimes \chi_D \) yields

\[ S_{\alpha}(A'B'T_B'D)_\sigma \geq \log |T_A'| + S_{\beta}(A'B')_\rho + \frac{2\alpha}{1 - \alpha} \log F. \]
Proof idea

- Both bounds together:
  
  \[
  \frac{2\alpha}{1 - \alpha} \log F \leq \log |Q| + \log |T_A| - \log |T'_A| - S_\beta(AB)_\rho + S_\alpha(B)_\rho
  \]

- \( \log |Q| \longleftrightarrow \text{quantum communication cost (for } n = 1) \)

- \( \log |T_A| - \log |T'_A| \longleftrightarrow \text{entanglement cost (for } n = 1) \)

- \( n \text{ copies of } \rho_{ABC} : \)
  
  \[
  \frac{2\alpha}{1 - \alpha} \log F_n \leq \log |Q| + \log |T_A| - \log |T'_A| \\
  - S_\beta(A^n B^n)_{\rho^\otimes n} + S_\alpha(B^n)_{\rho^\otimes n} \\
  = n(q_n + e_n - S_\beta(AB)_\rho + S_\alpha(B)_\rho)
  \]
1. Weak vs. strong converse

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More strong converse theorems

Using the Rényi entropy method, we derive strong converse theorems for the following protocols (details in [arXiv:1506.02635]):

1. State redistribution with feedback (allowing quantum communication from Bob to Alice)
2. Coherent state merging and quantum state splitting (special cases of state redistribution)
3. Measurement compression with quantum side information (QSI)
4. Randomness extraction against QSI
5. Data compression with QSI
Open questions

Applications of the fidelity bound

- Crucial mathematical result in the proofs:

\[
\tilde{S}_\alpha(A|B)_\rho - \tilde{S}_\beta(A|B)_\sigma \geq \frac{2\alpha}{1 - \alpha} \log F(\rho_{AB}, \sigma_{AB})
\]

where \( \alpha \in (1/2, 1) \) and \( \beta = \alpha/(2\alpha - 1) \).

- Can we use this to derive strong converse theorems for other protocols?

- Example: Strong converse for degradable channels

- In entanglement generation, fidelity bound yields a bound on figure of merit in terms of entanglement generation capacity and a “Rényi coherent information”.

- Additivity result or other estimates for this quantity needed to infer strong converse theorem.


Thank you very much!