Classical Statistical Mechanics of Ideal Gases

Definition of ideal gas

A gas is <u>ideal</u> if each of its many-particle states is a $\left\{ \begin{matrix} \text{symmetrized} & \text{(bosons)} \\ \text{antisymmetrized} & \text{(fermions)} \end{matrix} \right\}$ product of single-particle states.

Physically, this is true if particles interact so weakly that their eigenstates are effectively those of isolated particles. Still, the particles interact to the extent that they can exchange energy and other conserved properties. The definition of ideal admits the possibility that particle an (anti-) exclusion principle.

The basic simplifying feature of an ideal gas is that its 1-particle states may be treated as independent subsystems.

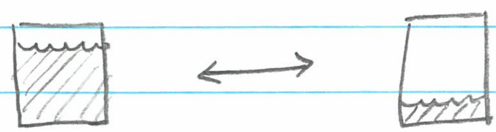
Microscopic reversibility and thermodynamic equilibrium

Microscopic reversibility, or T symmetry, asserts that the rate (transition probability per unit time) for one state A to convert into another state B equals the inverse rate

$$\text{rate } (A \rightarrow B) = \text{rate } (A \leftarrow B).$$

Consider

A B



Because rates $A \rightleftarrows B$ are equal, after a long time both states are occupied with equal probability

A B

Condition of all states being occupied with equal probability, subject to conservation laws, is the condition of Thermodynamic Equilibrium (TE).

It is ironic that the irreversible approach to TE is a consequence of microscopic reversibility.

Comment.

There are T-violating processes involving oscillations between 3 generations.

But CPT symmetry

charge time

parity

requires that rates be equal for

$$\begin{tikzcd} A \arrow[rr] & & B \arrow[dl] \\ & C \arrow[ul] & \end{tikzcd} \qquad \begin{tikzcd} \bar{A} \arrow[dr] & & \bar{B} \arrow[ll] \\ & \bar{C} \arrow[ur] & \end{tikzcd}$$

When is TE true?

Consider

$$A \rightleftarrows B .$$

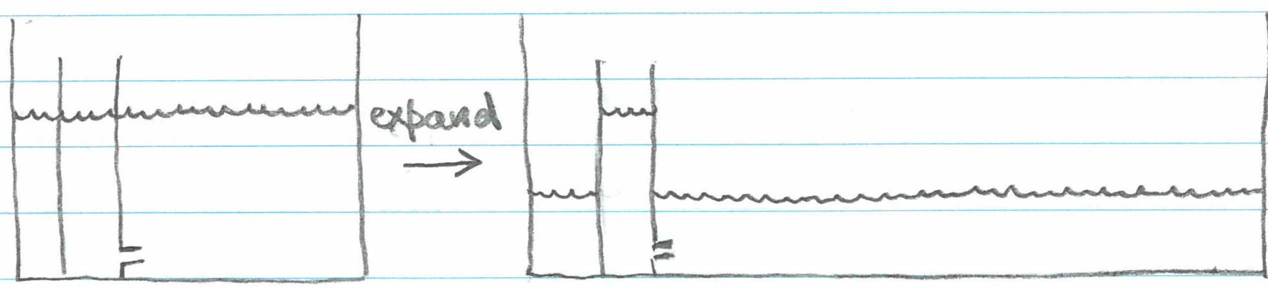
Then A & B will be in TE (equally probable) if the rate $A \leftrightarrow B$ is faster than any other processes connecting A or B to other states of the system.

$$\begin{tikzcd} A \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] & B \arrow[d, leftrightarrow] \\ C \arrow[ur, Leftrightarrow] \arrow[r, leftrightarrow] & D \end{tikzcd}$$

Here A & B come into TE,
then A, B, C " " TE,
then A, B, C, D " " TE.

Expansion drives the Universe out of TE

Universe post-inflation is dense, hot, mostly in TE.
As it expands, density decreases,
temperature decreases, rates slow,
processes freeze out.

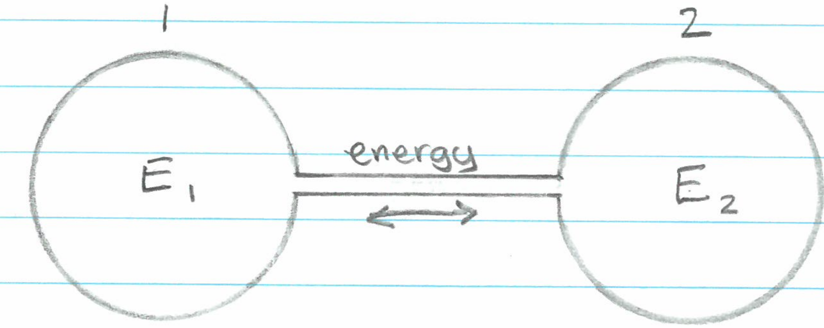


Freeze out:

1. Baryons — Baryogenesis
2. WIMPs (= CDM?)
   Weakly Interacting Massive Particles
3. Neutrons } Nucleosynthesis
4. Nuclei }
5. Photons, atoms — Recombination

Temperature T

Consider two systems allowed to exchange energy.



$$E_1 + E_2 = E = \text{constant}.$$

The number of states $\Omega(E_1, E_2)$ of the entire setup is, by ideality, a product of the number of states of each part

$$\Omega(E_1, E_2) = \Omega_1(E_1)\, \Omega_2(E_2).$$

In TE, all states Ω are equally occupied, and the energy E may be distributed arbitrarily between the two parts E_1 and E_2. But not all distributions of energy are equally likely. The most probable distribution of energy is where

$$\frac{\partial \ln \Omega(E_1, E_2)}{\partial E_1} = 0 \quad \text{subject to } E_1 + E_2 = E$$

ie. $$\frac{\partial \ln \Omega_1(E_1)}{\partial E_1} - \frac{\partial \ln \Omega_2(E_2)}{\partial E_2} = 0$$

ie $$\frac{\partial \ln \Omega_1}{\partial E_1} = \frac{\partial \ln \Omega_2}{\partial E_2} = \frac{1}{T} \qquad \text{units } k = 1$$

$\uparrow$ Boltzmann's constant

is same for both systems.

More generally

$$\boxed{\frac{\partial \ln \Omega}{\partial E} = \frac{1}{T}}$$

is same between multiple systems in mutual TE allowed to exchange energy.

Other thermodynamic variables

Same argument applies to systems in TE allowed to exchange any other conserved quantity C. To each conserved quantity C is associated a thermodynamic variable y

$$\frac{\partial \ln \Omega}{\partial C} = y$$

which is the same between multiple systems in mutual TE.

Conserved quantity C		$\partial \ln \Omega / \partial C$
Energy	E	$1/T$
Number	N_x	$-\mu/T$
Volume	V	p/T
Momentum	$\vec{P}$	$-\vec{v}/T$
Angular momentum	$\vec{J}$	$-\vec{\omega}/T$

Volume V is an externally imposed parameter rather than a conserved quantity.

Entropy S (provisional)

$$\boxed{S \equiv \ln \Omega}$$

Note this definition is provisional: it is valid only for a closed system in TE.

Nevertheless, this definition already implies:
1. Entropy is additive over independent subsystems.
2. If you put 2 systems together, their entropy increases.
3. Entropy is a maximum in TE.
4. Entropy is zero if there is only one state.

The two fundamental equations of thermodynamics

$$\boxed{dS = \frac{1}{T}\,dU - \sum_x \frac{\mu_x}{T}\,dN_x + \frac{p}{T}\,dV}$$

where the internal energy U is the energy in the rest (possibly rotating) frame of the fluid

$$\underset{\substack{\uparrow \\ \text{internal} \\ \text{energy}}}{dU} = \underset{\substack{\uparrow \\ \text{total} \\ \text{energy}}}{dE} - \underset{\substack{\uparrow \\ \text{bulk} \\ \text{velocity}}}{\vec{v}} \cdot d\underset{\substack{\uparrow \\ \text{momentum} \\ \text{of center} \\ \text{of mass}}}{\vec{P}} - \underset{\substack{\uparrow \\ \text{bulk} \\ \text{angular} \\ \text{velocity}}}{\vec{\omega}} \cdot d\underset{\substack{\uparrow \\ \text{angular} \\ \text{momentum} \\ \text{about center} \\ \text{of mass}}}{J}$$

In TE, the parts of a system should not only have the same temperature T and chemical potential(s) μ_x, but should move with the same bulk velocity $\vec{v}$, and should rotate at uniform angular velocity $\vec{\omega}$.

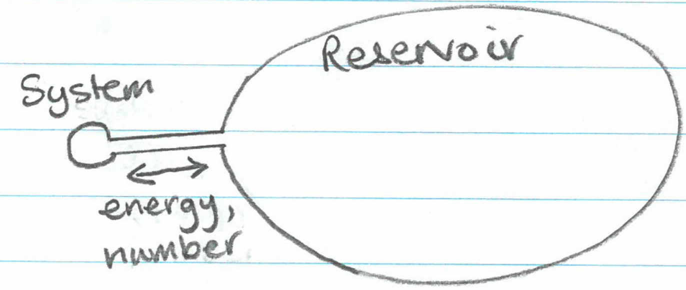
In an ideal gas, S, U, N_x are all proportional to volume V at fixed T, μ_x, p. Hence, in an ideal gas

$$\boxed{S = \frac{U}{T} - \sum_x \frac{\mu_x}{T} N_x + \frac{p V}{T}} .$$

State occupation probabilities

So far considered large, closed system. But what about individual single particle states? Need to consider not just the most probable condition, but the full probability distribution.

Consider small system attached to huge reservoir



$$E = E_{sys} + E_{res}$$
$$N = N_{sys} + N_{res}$$

What is the probability P_j that system is in a specific state j?

In TE, all states of the system plus reservoir equally occupied.

Therefore
$P_j \propto$ (# states of system such that system is in state j)
$\times$ (# states of reservoir such that system is in state j).

If state j has energy E_j and number N_j, then the reservoir must have energy $E - E_j$ and number $N - N_j$ (and is otherwise unconstrained).

So $P_j \propto 1 \times \Omega_{res}(E - E_j, N - N_j)$

But $E_j \ll E$, $N_j \ll N$, so Taylor expand:

$$\ln P_j = \ln P_0 - E_j \overset{\frac{1}{T}}{\cancel{\frac{\partial \ln \Omega_{res}}{\partial E_{res}}}} - N_j \overset{-\frac{\mu}{T}}{\cancel{\frac{\partial \ln \Omega_{res}}{\partial N_{res}}}}$$
$$+ \frac{E_j^2}{2!} \frac{\partial^2 \ln \Omega}{\partial E^2} + E_j N_j \frac{\partial^2 \ln \Omega}{\partial E \partial N} + \frac{N_j^2}{2!} \frac{\partial^2 \ln \Omega}{\partial N^2} - \ldots$$

For an ideal gas, $\ln \Omega \propto V$, and $E, N \propto V$
so $\frac{\partial^2 \ln \Omega}{\partial E^2} \propto V^{-1} \rightarrow 0$ as $V \rightarrow \infty$

so 2nd order and higher terms in Taylor expansion can be neglected.

Thus

$$\ln P_j = \ln P_0 - \frac{E_j}{T} + \frac{\mu N_j}{T}$$

or

$$\boxed{P_j \propto \exp\left(\frac{-E_j + \mu N_j}{T}\right)}$$

Single-particle state occupation numbers f

Let j be a single-particle state. Drop index j.

A single-particle state may contain

$$\begin{cases} 0 \text{ or } 1 \text{ particle} & (\text{fermions}) \\ 0, 1, 2, \ldots \text{ particles} & (\text{bosons}) \end{cases}$$

The probability that the state contains N particles is $P(N) \propto \exp\left(\frac{-EN + \mu N}{T}\right) \propto p^N$

The occupation number f is the mean number of particles in the state

$$f = \frac{\sum_N N p^N}{\sum_N p^N}$$

Fermions

$\sum_{N=0}^{1}$:

$$\boxed{f_{FD} = \frac{1}{e^{\frac{E-\mu}{T}} + 1}}$$

$\uparrow$ Fermi-Dirac

Bosons

$$\sum_{N=0}^{\infty} : \quad \boxed{f_{BE} = \frac{1}{e^{\frac{E-\mu}{T}} - 1}}$$

$\uparrow$ Bose-Einstein

Boltzmann

Limit of small occupation numbers, $f \ll 1$,
ie. $\mu \rightarrow$ large negative

$$\boxed{f_{\text{Boltzmann}} = e^{\frac{\mu - E}{T}}}$$

Entropy S again

In closed system (fixed E, N) in TE,
all states j of system have equal probability,

so $\quad P_j = \dfrac{1}{\Omega} \leftarrow$ number of states

so $\quad S \equiv \ln \Omega = -\ln P_j$

In a subsystem, the probability $P_j \propto e^{\frac{-E_j + \mu N_j}{T}}$
of the subsystem being in state j varies.
To preserve additivity over independent subsystems, define entropy S by

$$\boxed{S \equiv - \sum_{\text{states } j} P_j \ln P_j} \; .$$

This definition is completely general.
Earlier equations of thermodynamics remain valid.

Equilibrium number densities for Boltzmann species

Consider Boltzmann ($f \ll 1$) species.
Number density n is

$$n = \int f \, \underset{\uparrow}{g} \frac{d^3p}{(2\pi)^3}$$

number of spin states

$$= e^{\mu/T} \int e^{-E/T} g \frac{d^3p}{(2\pi)^3}$$

$$= e^{\mu/T} n^{(0)}(T)$$

where, for non-relativistic species,

$$\boxed{n^{0}(T) = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T}}$$

TE condition on chemical potentials μ can thus be re-expressed as a condition on number densities. For example

$$p + e \longleftrightarrow H + \gamma$$

in TE satisfies

$$\mu_p + \mu_e = \mu_H \qquad (\text{note } \mu_\gamma = 0)$$

which may be re-expressed as

$$\frac{n_p}{n_p^{(0)}} \frac{n_e}{n_e^{(0)}} = \frac{n_H}{n_H^{(0)}}$$

or

$$\frac{n_p n_e}{n_H} = \frac{n_p^{(0)} n_e^{(0)}}{n_H^{(0)}}$$

$$= \frac{g_p g_e}{g_H}\left(\frac{m_p m_e}{m_H} \frac{T}{2\pi}\right)^{3/2} e^{\frac{-m_p - m_e + m_H}{T}} \leftarrow -\chi_H$$

ionization energy

Free particle distributions

$$\boxed{\underset{\substack{\uparrow \\ \text{mean number} \\ \text{of particles}}}{dN} = \underset{\substack{\uparrow \\ \text{occupation} \\ \text{number}}}{f} \; \underset{\substack{\uparrow \\ \text{number of} \\ \text{spin states}}}{g} \; \underset{\substack{\uparrow \\ \text{number of} \\ \text{free-particle} \\ \text{eigenstates}}}{\frac{d^3r\, d^3p}{(2\pi\hbar)^3}}}$$

will routinely set $\hbar = 1$

Rate coefficients

Consider reaction

$$1 + 2 \longleftrightarrow 3 + 4$$

$$\text{Rate}\,(\rightarrow) \propto f_1 f_2 (1 \mp f_3)(1 \mp f_4)$$

$$\text{Rate}\,(\leftarrow) \propto f_3 f_4 (1 \mp f_1)(1 \mp f_2)$$

$1 - f$: Fermi-Dirac blocking factor

$1 + f$: Bose-Einstein stimulation factor.

Should be that rates balance in TE.

Check:

$$f_1 f_2 (1 \mp f_3)(1 \mp f_4) \underset{TE}{\overset{?}{=}} f_3 f_4 (1 \mp f_1)(1 \mp f_2)$$

true iff

$$\frac{f_1}{1 \mp f_1} \frac{f_2}{1 \mp f_2} = \frac{f_3}{1 \mp f_3} \frac{f_4}{1 \mp f_4} .$$

But $\dfrac{f}{1 \mp f} = e^{\frac{-E+\mu}{T}}$

so true iff

$$\frac{-E_1+\mu_1}{T} + \frac{-E_2+\mu_2}{T} = \frac{-E_3+\mu_3}{T} + \frac{-E_4+\mu_4}{T}$$

which is true in TE because

$$E_1 + E_2 = E_3 + E_4$$

$$\mu_1 + \mu_2 = \mu_3 + \mu_4$$

In Boltzmann case, $f \ll 1$, blocking / stimulation factors $1 \mp f \approx 1$ can be neglected.

Boltzmann equation

Boltzmann equation describes evolution of the occupation number f

$$\boxed{\frac{df}{dt} = \underbrace{C[f]}_{\text{collision term}}}$$

In the absence of collisions, $C[f] = 0$, the occupancy of each state remains unchanged (hey, they are eigenstates, right?).

In FRW universe, write $f(\underbrace{t}_{\text{cosmic time}}, \underbrace{\vec{x}}_{\text{comoving position}}, \underbrace{\vec{p}}_{\text{proper momentum}})$

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\vec{x}}{dt} \cdot \frac{\partial f}{\partial \vec{x}} + \frac{d\vec{p}}{dt} \cdot \frac{\partial f}{\partial \vec{p}}$$

4

THE BOLTZMANN EQUATIONS

We are interested in the anisotropies in the cosmic distribution of photons and inhomogeneities in the matter. Figure 4.1 shows why these are complicated to calculate. The photons are affected by gravity and by Compton scattering with free electrons. The electrons are tightly coupled to the protons. Both of these, of course, are also affected by gravity. The metric which determines the gravitational forces is influenced by all these components plus the neutrinos and the dark matter. Thus to solve for the photon and dark matter distributions, we need to simultaneously solve for all the other components.

There is a systematic way to account for all of these couplings. We write down a Boltzmann equation for each species in the universe. We have already encountered the Boltzmann equation in its integrated form in Chapter 3. There we were interested solely in the number density of the dark matter, the neutrons, and the free electrons. The number density is the integral over all momenta of the distribution function. Here we will be interested in more detailed information, not just the integrated number density, but the full distribution of photons, say, as a function of momentum. We then need a more primitive version of Eq. (3.1). Schematically, the unintegrated Boltzmann equation is

$$\frac{df}{dt} = C[f]. \tag{4.1}$$

The right-hand side of the Boltzmann equation contains all possible collision terms. These terms in general are complicated functionals of the distribution functions of the various components. In the absence of collisions, the distribution function obeys $df/dt = 0$. This seemingly innocent equation says that the number of particles in a given element of phase space does not change with time. The catch is that the phase space elements themselves are moving in time in complicated ways due to the nontrivial metric. This catch makes the problem more difficult than it seems from Eq. (4.1). Nonetheless, we can still progress systematically by reexpressing the full derivative in terms of partial derivatives.

contact with the plasma. In succeeding chapters, we will move beyond uniformity and explore temperatures which depend on both position and direction of propagation.

3.1 BOLTZMANN EQUATION FOR ANNIHILATION

proper

The Boltzmann equation formalizes the statement that the rate of change in the abundance of a given particle is the difference between the rates for producing and eliminating that species. Suppose that we are interested in the number density n_1 of species 1. For simplicity, let's suppose that the only process affecting the abundance of this species is an annihilation with species 2 producing two particles, imaginatively called 3 and 4. Schematically, $1+2 \leftrightarrow 3+4$; i.e., particle 1 and particle 2 can annihilate producing particles 3 and 4, or the inverse process can produce 1 and 2. The Boltzmann equation for this system in an expanding universe is

comoving

$$a^{-3}\frac{d\left(n_1 a^3\right)}{dt} = \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} \int \frac{d^3p_3}{(2\pi)^3 2E_3} \int \frac{d^3p_4}{(2\pi)^3 2E_4}$$

$\hbar = 1$ $\quad g_{spin}$?

$$\times\ (2\pi)^4 \delta^3(p_1 + p_2 - p_3 - p_4)\delta(E_1 + E_2 - E_3 - E_4)\underbrace{|\mathcal{M}|^2}_{\text{dimensionless}}$$

$$\times\ \{f_3 f_4[1 \pm f_1][1 \pm f_2] - f_1 f_2[1 \pm f_3][1 \pm f_4]\}\,. \tag{3.1}$$

In the absence of interactions, the left-hand side of Eq. (3.1) says that the density times the scale factor cubed is conserved. This reflects the nature of the expanding universe: as the comoving grid expands, the volume of a region containing a fixed number of particles grows as a^3. Therefore, the physical number density of these particles falls off as a^{-3}. Interactions are included in the right-hand side of the Boltzmann equation. Let's consider the interaction term starting from the last line and moving up. Putting aside the $1 \pm f$ terms on the last line, we see that the rate of producing species 1 is proportional to the occupation numbers of species 3 and 4, f_3 and f_4. Similarly the loss term is proportional to $f_1 f_2$. The $1 \pm f$ terms, with plus sign for bosons such as photons and minus sign for fermions such as electrons, represent the phenomena of Bose enhancement and Pauli blocking. If particles of type 1 already exist, a reaction producing more such particles is more likely to occur if 1 is a boson and less likely if a fermion. I have suppressed the momentum dependence of f, but of course all the occupation numbers depend on the corresponding momentum (e.g., $f_1 = f_1(p_1)$). Moving upward, the Dirac delta functions on the second line in Eq. (3.1) enforce energy and momentum conservation; the factors of 2π are the result of moving from discrete Kronecker δ's to the continuous Dirac version. The energies here are related to the momenta via $E = \sqrt{p^2 + m^2}$. The amplitude on the second line $\mathcal{M}$ is determined from the fundamental physics in question. For example, if we were interested in the Compton scattering of el

The proper number density $n(t,\vec{x})$ is

$$n(t,\vec{x}) = \int f(t,\vec{x},\vec{p})\,\frac{d^3p}{(2\pi)^3}$$

Show that in FRW universe

$$\boxed{\int \frac{df}{dt}\frac{d^3p}{(2\pi)^3} = a^{-3}\frac{\partial(a^3 n)}{\partial t}}$$

Hint: use $\frac{\partial f}{\partial \vec{x}}$ = constant, $\vec{p} \propto a^{-1}$.

Collision term

For example, collision term for process

$$1 + 2 \longleftrightarrow 3 + 4$$

gives, for evolution of species 1,

$$\boxed{\begin{aligned}\int C[f_1]\frac{d^3p_1}{(2\pi)^3} = \int &\frac{d^3p_1}{2E_1(2\pi)^3}\frac{d^3p_2}{2E_2(2\pi)^3}\frac{d^3p_3}{2E_3(2\pi)^3}\frac{d^3p_4}{2E_4(2\pi)^3} \\ &\times (2\pi)^4\delta_D^4(p_1+p_2-p_3-p_4) \\ &\times \left[-f_1f_2(1\mp f_3)(1\mp f_4) + f_3f_4(1\mp f_1)(1\mp f_2)\right] \\ &\times |\mathcal{M}|^2\end{aligned}}$$

whose various parts come from:

$$\frac{d^3p}{2E(2\pi)^3}$$

Manifestly Lorentz invariant

$$\int_E \frac{d^4p}{(2\pi)^4}\cdot 2\pi\delta_D(E^2-p^2-m^2) \qquad p \equiv (E,\vec{p})$$

$$= \int_E \frac{d^3p}{(2\pi)^3}\frac{dE}{2\pi}\,2\pi\,\delta_D(E^2-p^2-m^2) = \frac{d^3p}{2E(2\pi)^3}.$$

$(2\pi)^4 \delta_D^4 (p_1 + p_2 - p_3 - p_4)$

Imposes energy-momentum conservation on the process $1+2 \leftrightarrow 3+4$.

Same as

$$(2\pi)^3 \delta_D^3 (\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\, 2\pi \delta_D (E_1 + E_2 - E_3 - E_4).$$

$-f_1 f_2 (1 \mp f_3)(1 \mp f_4) + f_3 f_4 (1 \mp f_1)(1 \mp f_2)$

Expresses fact that rates are proportional to the occupancy of initial states, modulated by blocking/stimulation factors of final states.

$|\mathcal{M}|^2$

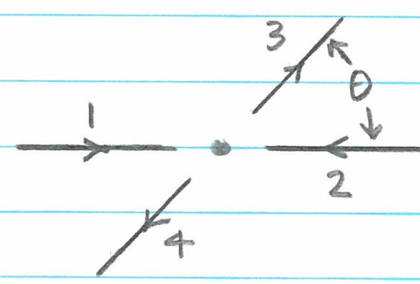
A Lorentz-invariant transition probability (amplitude squared), computed typically by quantum field theory (Feynmann diagrams).

$|\mathcal{M}|^2$ is the thing that is unchanged for example under T symmetry.

In absence of spin effects, $|\mathcal{M}|^2$ depends on center-of-mass energy

$$-(p_1 + p_2)^2$$

and scattering angle θ.



$|\mathcal{M}|^2$ happens to be dimensionless for $1+2 \leftrightarrow 3+4$ process.

Big Bang Nucleosynthesis

Overview

1. n/p ratio determined by freeze out of weak interactions at $T \sim 1$ MeV.

2. Almost all n subsequently incorporated into ^4He at $T \sim .07$ MeV, so $^4He/H$ ratio is determined by n/p.

3. Light elements D, ^3He, ^7Li synthesized in small amounts, but no heavy elements, because of absence of stable nuclei of mass 5 or 8.

Photons γ

Electromagnetic interactions

$$e + \bar{e} \leftrightarrow \gamma + \gamma$$

keep electrons e, positrons $\bar{e}$, and photons γ in mutual TE.

Photon number is not conserved (photons are their own antiparticles)

so $\boxed{\mu_\gamma = 0}$.

Thus $\quad f_\gamma = \dfrac{1}{e^{E/T} - 1}$ $\quad$ Planck distribution.

Electrons e and positrons $\bar{e}$

In TE with photons

$$e + \bar{e} \longleftrightarrow \gamma + \gamma$$

$$\Rightarrow \quad \mu_e + \mu_{\bar{e}} = 0$$

so $\quad \boxed{\mu_{\bar{e}} = -\mu_e}$

Initially ($T \gg m_e = .511$ MeV) $e\bar{e}$ are relativistic, in which case $n_e \approx n_{\bar{e}} \approx \underbrace{\frac{3}{4}}_{\text{fermion/boson factor}} n_\gamma$.

As T decreases, $e\bar{e}$ annihilate, leaving $\quad n_e \approx n_b \approx 10^{-9}\, n_\gamma$

$$n_{\bar{e}} \approx 0.$$

If it were true that $n_e = n_{\bar{e}}$, then $\mu_e = \mu_{\bar{e}} = 0$, but in reality $n_e > n_{\bar{e}}$, so $\mu_e > \mu_{\bar{e}}$.

Relativistic: $\quad f_e \approx f_{\bar{e}} \approx \dfrac{1}{e^{E/T} + 1} \qquad E \approx p$.

Non-relativistic: $\quad f_e \approx e^{\mu/T} e^{-E/T} \qquad E \approx m + \dfrac{p^2}{2m}$.

Neutrinos ν and antineutrinos $\bar{\nu}$

While $e\bar{e}$ are relativistic (hence abundant)

$$\gamma + e \longleftrightarrow \gamma + e$$

and similar keep $\nu\bar{\nu}$ in TE with $e\bar{e}$.
$\nu\bar{\nu}$ decouple at $T \sim 1\,\mathrm{MeV}$, but retain relativistic TE distribution

$$f_\nu \approx f_{\bar{\nu}} \approx \frac{1}{e^{E/T_\nu} + 1}$$

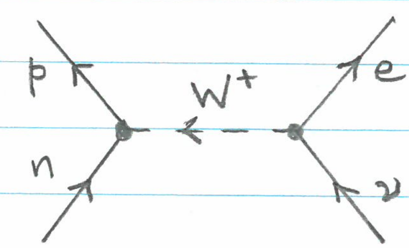
Neutrons n and protons p

Weak interactions, which transmute n and p into each other, freeze out at $T \sim 1\,\mathrm{MeV}$.

Transmuting interactions:

① $n + \nu \longleftrightarrow p + e$
② $n + \bar{e} \longleftrightarrow p + \bar{\nu}$
③ $n \longleftrightarrow p + e + \bar{\nu}$
④ $n + \bar{e} + \nu \longleftrightarrow p$

all have the same Feynmann diagram



and so have the same transition probability

$$|\mathcal{M}|^2 = 32\, G_F^2 \,(1 + 3 g_A^2)\, m_p^2\, p_\nu\, p_e$$

G_F: Fermi constant; g_A: axial-vector coupling

"Antimatter: matter with negative mass going backwards in time looks like positive mass going forwards in time."

— Feynmann

$$\uparrow \bar{e} \;=\; \downarrow e$$

Consider for example reaction ①.
The occupancy factors in the rate are

$$- f_n(\mu_n, T)\, f_\nu(T)\,[1 - f_e(T)]$$
$$+ f_p(\mu_p, T)\, f_e(T)\,[1 - f_\nu(T)]$$

at least while $e\bar{e}$ are relativistic.

Electromagnetic interactions keep n and p distributions thermal, with $T_n = T_p = T_\gamma$.
$m_n \approx m_p \approx 1\,\text{GeV} \Rightarrow$ non-relativistic.
$n_n \sim n_p \sim n_b \sim 10^{-9} n_\gamma \Rightarrow$ Boltzmann
so

$$f_n = e^{\mu_n/T}\, e^{-E_n/T}$$
$$f_p = e^{\mu_p/T}\, e^{-E_p/T}$$

Hence

$$a^{-3}\frac{d(a^3 n_n)}{dt} = -n_n \lambda_{\rightarrow}(T) + n_p \lambda_{\leftarrow}(T)$$

with

$$\frac{\lambda_{\rightarrow}(T)}{\lambda_{\leftarrow}(T)} = \left.\frac{n_n}{n_p}\right|_{TE} = \frac{n_n^{(0)}(T)}{n_p^{(0)}(T)}$$
$$= \frac{g_n}{g_p}\, e^{\frac{-m_n + m_p}{T}} = e^{-Q/T}.$$

$Q \equiv m_n - m_p = 1.293\,\text{MeV}$

So

$$\boxed{a^{-3}\frac{d(a^3 n_n)}{dt} = \lambda_{\rightarrow}(T)\left(-n_n + n_p e^{-Q/T}\right)}$$

This was derived for reaction ①, but all reactions ① – ④ take the same form, modifying only $\lambda_{\rightarrow}(T)$.

Define $X_n \equiv \dfrac{n_n}{n_n + n_p} = \dfrac{n_n}{n_b}$.

Recall $dt = \dfrac{da}{aH}$.

Universe is radiation-dominated, so $H \propto a^{-2}$.

And $T \propto \dfrac{1}{a}$, neglecting departure caused by $e\bar{e}$ annihilation.

Set $a = \dfrac{Q}{T}$ (ie normalize $a = 1$ at $Q = T$)

Then

$$\frac{dX_n}{da} = \frac{a\,\lambda_{\rightarrow}(Q/a)}{H(Q)}\left[-X_n + (1 - X_n)e^{-a}\right]$$

See Dodelson Ch. 3 p. 65–68 and Ex. 3.

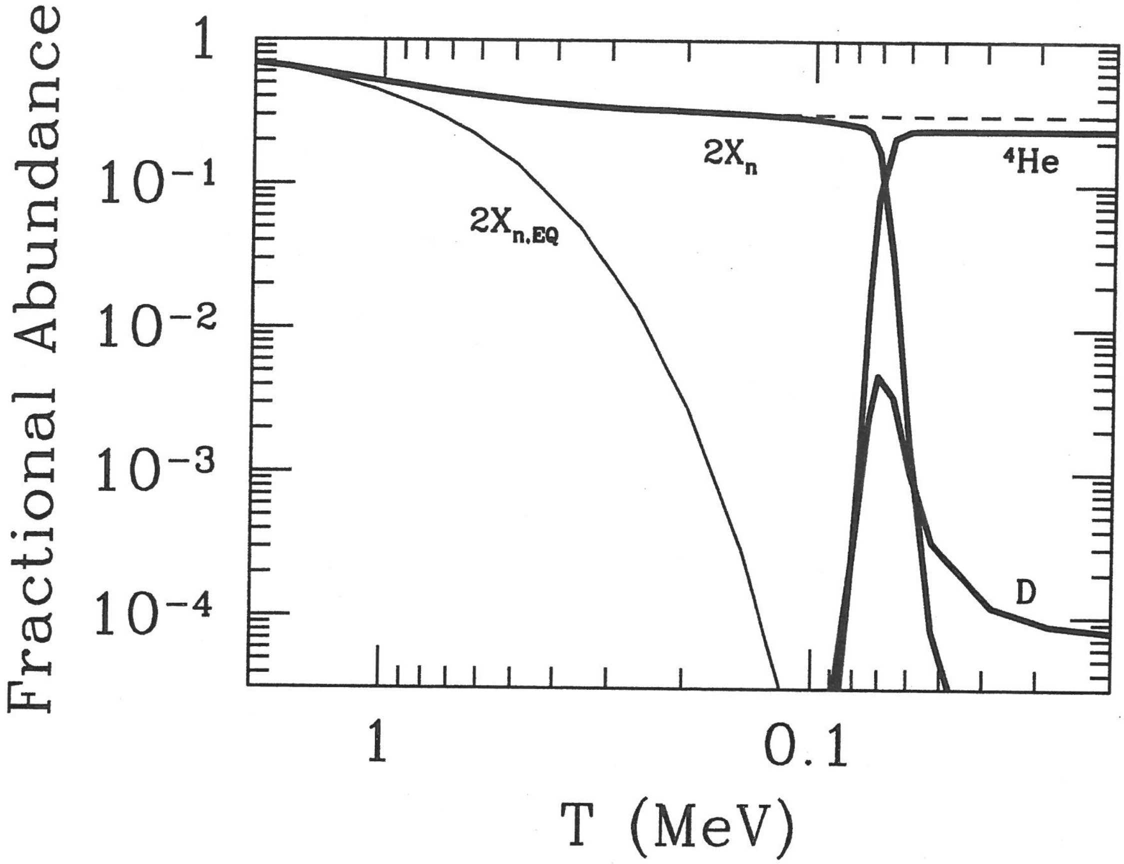


Figure 3.2. Evolution of light element abundances in the early universe. Heavy solid curves are results from Wagoner (1973) code; dashed curve is from integration of Eq. (3.27); light solid curve is twice the neutron equilibrium abundance. Note the good agreement of Eq. (3.27) and the exact result until the onset of neutron decay. Also note that the neutron abundance falls out of equilibrium at $T \sim$MeV.

3.2.2 Light Element Abundances

A useful way to approximate light element production is that it occurs instantaneously at a temperature $T_{\rm nuc}$ when the energetics compensates for the small baryon to photon ratio. Let's consider deuterium production as an example, with Eq. (3.17) as our guide. The equilibrium deuterium abundance is of order the baryon abundance (i.e. if the universe stayed in equilibrium, all neutrons and protons would form deuterium) when Eq. (3.17) is of order unity, or

$$\ln(\eta_b) + \frac{3}{2}\ln(T_{\rm nuc}/m_p) \sim -\frac{B_D}{T_{\rm nuc}}. \tag{3.32}$$

Equation (3.32) suggests that deuterium production takes place at $T_{\rm nuc} \sim 0.07$ MeV, with a weak logarithmic dependence on η_b.

Since the binding energy of helium is larger than that of deuterium, the exponential factor $e^{B/T}$ favors helium over deuterium. Indeed, Figure 3.2 illustrates that helium is produced almost immediately after deuterium. Virtually all remain-

^{4}He mass fraction Y

To a good approximation, all n's subsequently go into ^{4}He.

Define ^{4}He mass fraction Y

$$\boxed{Y \equiv \frac{4 n_{^4He}}{n_H + 4 n_{^4He}}}$$

whereas

$$X_n \equiv \frac{n_n}{n_p + n_n} \qquad n_n = 2 n_{^4He}$$

$$= \frac{2 n_{^4He}}{n_H + 4 n_{^4He}} \qquad n_p = n_H + 2 n_{^4He}$$

so $\boxed{Y = 2 X_n}$

Baryon-photon ratio η_b

$$n_b = \frac{3 \Omega_b H_0^2}{8\pi G m_b} = 0.2249 \left(\frac{\Omega_b h^2}{0.020}\right)\left(\frac{m_b}{937\,\mathrm{MeV}}\right)^{-1} \mathrm{m}^{-3}$$

$$n_\gamma = \frac{2\zeta(3)}{\pi^2}\left(\frac{kT_0}{c\hbar}\right)^3 = 0.4105\, T_{2.725K}^3 \ \mathrm{mm}^{-3}$$

$$\boxed{\eta_b \equiv \frac{n_b}{n_\gamma} = 0.546 \times 10^{-9}\left(\frac{\Omega_b h^2}{0.020}\right)}$$

Why does ^{4}He mass fraction Y depend on η_b?

Higher η_b makes D production earlier, when X_n is higher because fewer free n's have decayed.

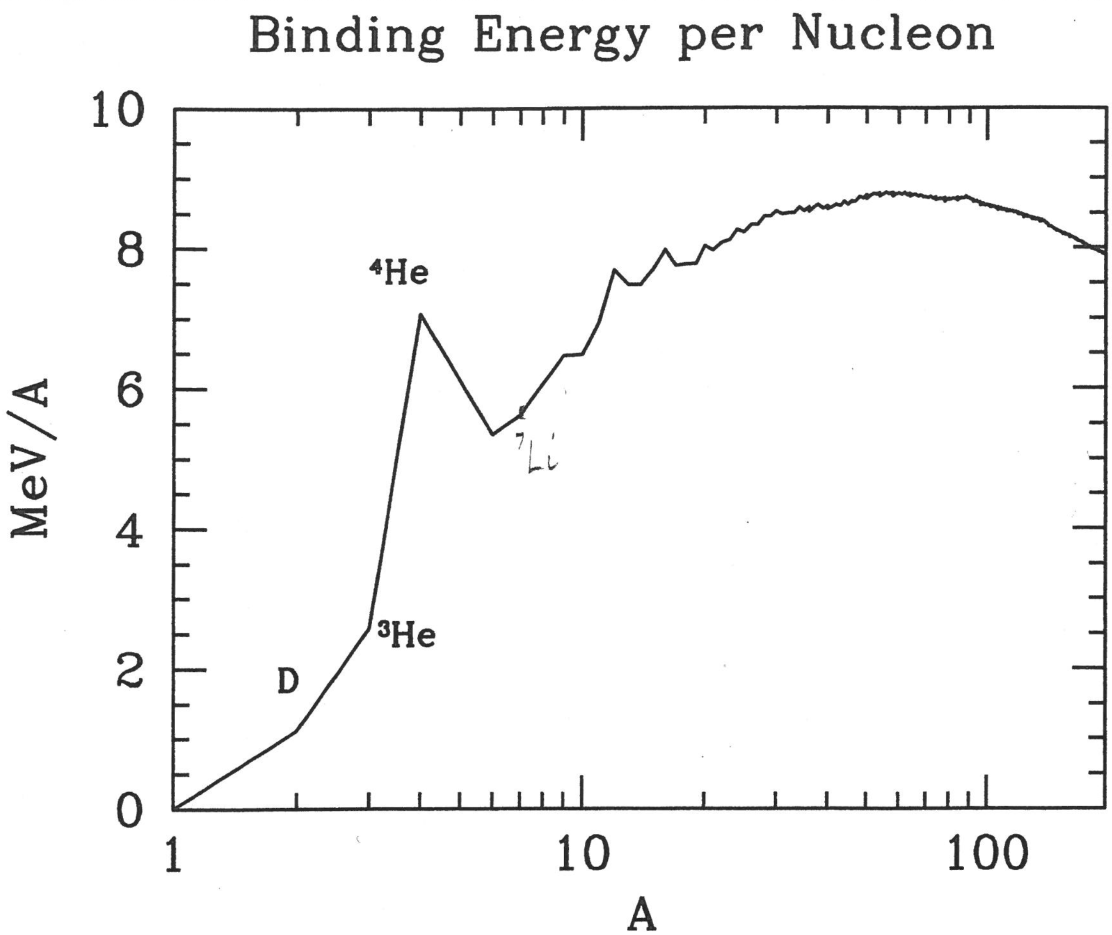


Figure 3.1. Binding energy of nuclei as a function of mass number. Iron has the highest binding energy, but among the light elements, ^{4}He is a crucial local maximum. Nucleosynthesis in the early universe essentially stops at ^{4}He because of the lack of tightly bound isotopes at $A = 5 - 8$. In the high-density environment of stars, three ^{4}He nuclei fuse to form ^{12}C, but the low baryon number precludes this process in the early universe.

Both of these simplifications — no heavy elements at all and only n/p above 0.01 MeV — rely on the physical fact that, at high temperatures, comparable to nuclear binding energies, any time a nucleus is produced in a reaction, it is destroyed by a high-energy photon. This fact is reflected in the fundamental equilibrium equation (3.10). To see how, let's consider this equation applied to deuterium production, $n + p \leftrightarrow \mathrm{D} + \gamma$. Since photons have $n_\gamma = n_\gamma^{(0)}$, the equilibrium condition becomes

$$\frac{n_D}{n_n n_p} = \frac{n_D^{(0)}}{n_n^{(0)} n_p^{(0)}}. \tag{3.14}$$

The integrals on the right, as given in Eq. (3.6), lead to

$$\frac{n_D}{n_n n_p} = \frac{3}{4} \left(\frac{2\pi m_D}{m_n m_p T} \right)^{3/2} e^{[m_n + m_p - m_D]/T}, \tag{3.15}$$

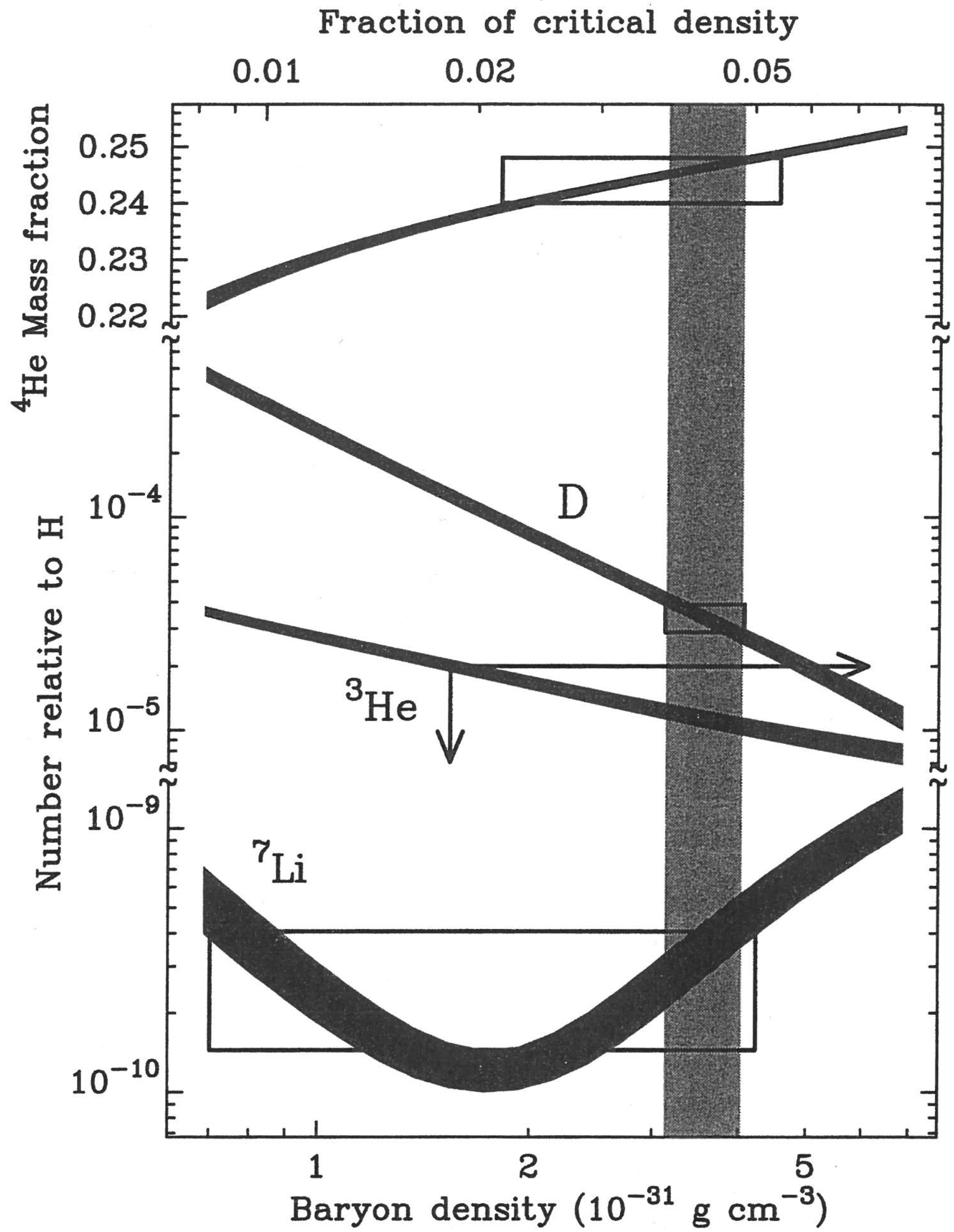


Figure 1.8. Constraint on the baryon density from Big Bang Nucleosynthesis (Burles, Nollett, and Turner, 1999). Predictions are shown for four light elements — ^{4}He, deuterium, ^{3}He, and lithium — spanning a range of 10 orders of magnitude. The solid vertical band is fixed by measurements of primordial deuterium. The boxes are the observations; there is only an upper limit on the primordial abundance of ^{3}He.

nary matter (baryons) contributes at most 5% of the critical density. Since the total matter density today is almost certainly larger than this — direct estimates give values of order 20–30% — nucleosynthesis provides a compelling argument for nonbaryonic dark matter.

The deuterium measurements (Burles and Tytler, 1998) are the new developments in the field. These measurements are so exciting because they explore the deuterium abundance at redshifts of order 3–4, well before much processing could

$\oplus\!\ominus$

Production of deuterium D

Deuterium D is produced mainly by

$$n + p \leftrightarrow D + \gamma$$

and destroyed by collisions with other nuclei ($p, n, D, T, {}^3He$).

$$a^{-3}\frac{d(a^3 n_D)}{dt} = n_n n_p \lambda_{\rightarrow}(T) - n_D \lambda_{\leftarrow}(T) - \text{other}.$$

Detailed balance implies

$$\frac{\lambda_{\leftarrow}(T)}{\lambda_{\rightarrow}(T)} = \left.\frac{n_n n_p}{n_D}\right|_{TE} = \frac{n_n^{(0)}(T)\, n_p^{(0)}(T)}{n_D^{(0)}(T)}$$

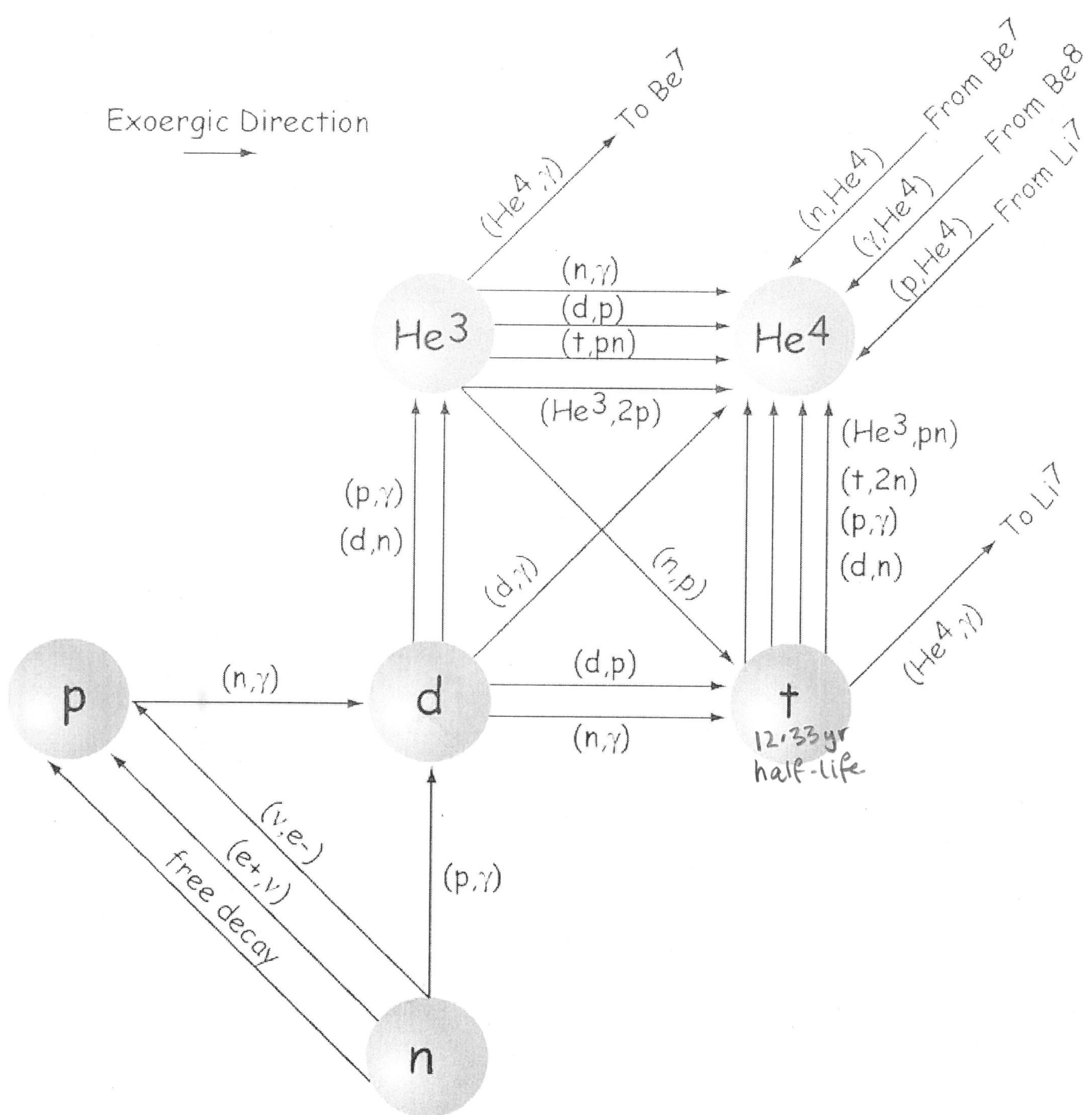
$$= \frac{g_n g_p}{g_D}\left(\frac{m_n m_p}{m_D}\frac{T}{2\pi}\right)^{3/2} e^{\frac{-m_p - m_n + m_D}{T}}$$

$$= \left(\frac{m_p T}{4\pi}\right)^{3/2} e^{-\underset{\uparrow\ 2.225\ \text{MeV}}{B_D}/T}$$

So

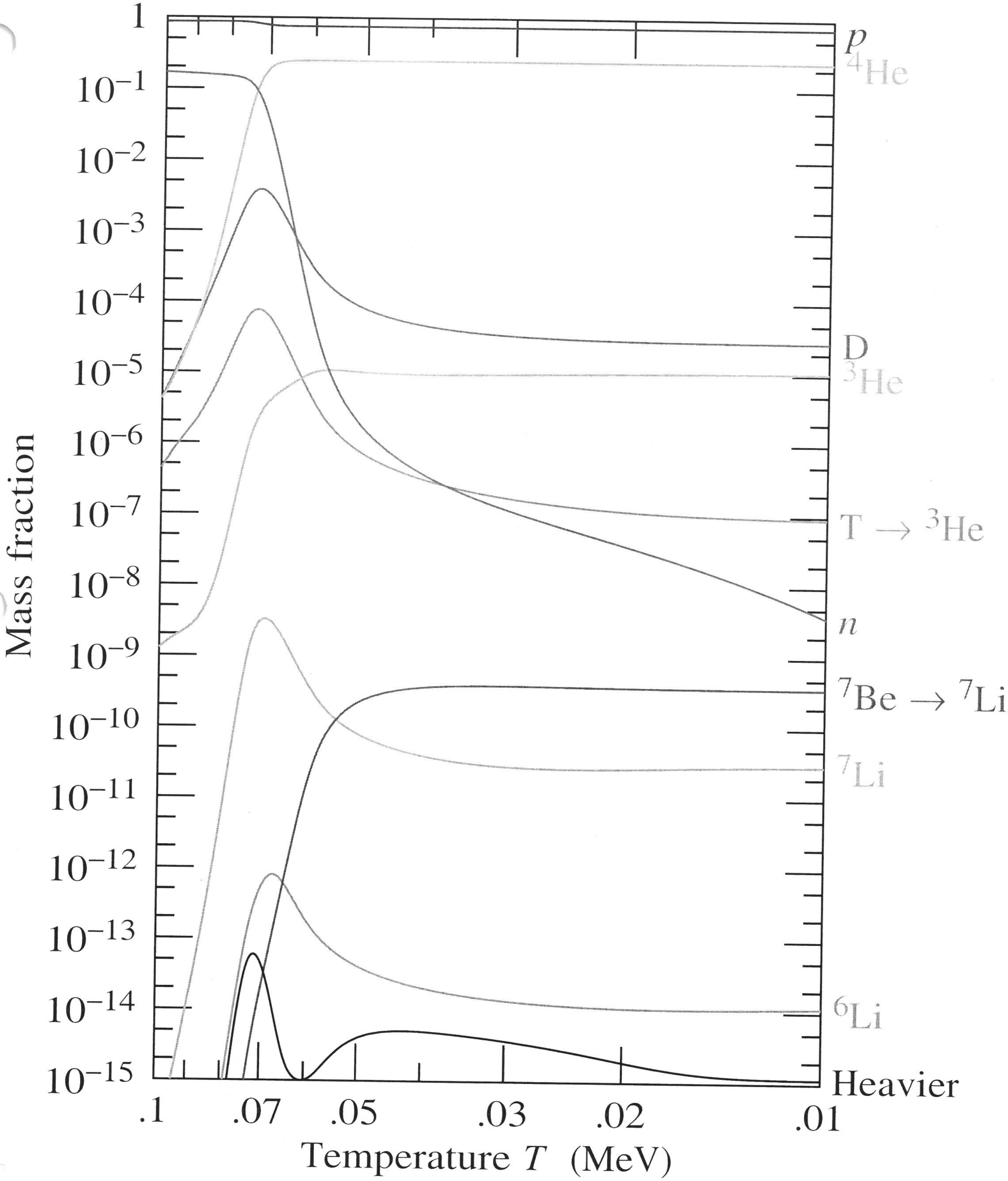
$$\boxed{a^{-3}\frac{d(a^3 n_D)}{dt} = \lambda_{\rightarrow}(T)\left[n_n n_p - n_D\left(\frac{m_p T}{4\pi}\right)^{3/2} e^{-B_D/T}\right] - \text{other}}$$

D is in TE with n and p for $T \gtrsim .08$ MeV.
T, ${}^3He$, ${}^4He$ feed off D, but rates are too slow to attain TE.



http://www.cococubed.com/code_pages/net_bigbang.shtml

$\Omega_b h^2 = 0.214 \qquad \tau_n = 885.8\,s$



Why does D abundance depend on η_b?

The higher the baryon/photon ratio η_b, the faster D is destroyed.

Absence of stable mass 5 or 8 nuclei

No stable nuclei of mass 5 or 8

$$^4\mathrm{He} + \left\{\begin{matrix} p \\ n \end{matrix}\right\} \not\rightarrow$$

$$^4\mathrm{He} + {}^4\mathrm{He} \not\rightarrow$$

BBN never creates heavy elements.

How are heavy elements created?

The triple-alpha reaction, a "3"-body process,

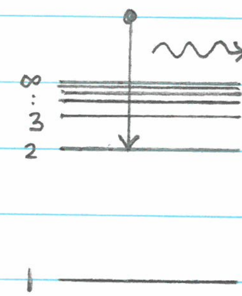
$$^4\mathrm{He} + {}^4\mathrm{He} \rightarrow \underset{\mu\text{s-lived}}{^8\mathrm{Be}}$$

$$^4\mathrm{He} + {}^8\mathrm{Be} \rightarrow {}^{12}\mathrm{C}^* \rightarrow {}^{12}\mathrm{C} + \gamma$$

occurs in stellar nucleosynthesis under conditions of very high density.

Recombination

For simplicity, consider only H, ignoring ^{4}He and other elements. Recombination occurs through



$$p + e \longleftrightarrow H^{(i)} + \gamma$$

$$a^{-3}\frac{d(a^3 n_p)}{dt} = -n_p n_e \sum_{i=1}^{\infty} \lambda_{rec}^{(i)}(T) + \sum_{i=1}^{\infty} n_{H^{(i)}} \lambda_{ion}^{(i)}(T)$$

Detailed balance

$$\Rightarrow \frac{\lambda_{ion}^{(i)}(T)}{\lambda_{rec}^{(i)}(T)} = \left.\frac{n_p n_e}{n_{H^{(i)}}}\right|_{TE} = \frac{n_p^{(0)}(T)\, n_e^{(0)}(T)}{n_{H^{(i)}}^{(0)}(T)}$$

$$= \frac{g_p g_e}{g_{H^{(i)}}} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\chi_{H^{(i)}}/T}$$

Collisional excitation and deexcitation

$$H^{(i)} + e \longleftrightarrow H^{(j)} + e + \gamma$$

keep levels $H^{(i)}$ in TE,

so $$\frac{n_{H^{(i)}}}{n_{H^{(1)}}} = \frac{n_{H^{(i)}}^{(0)}(T)}{n_{H^{(1)}}^{(0)}(T)} = e^{-\chi_{H^{(1)}} + \chi_{H^{(i)}}}$$

$$\chi_{H^{(1)}} = \chi_H = 13.6\,\text{eV}$$

So

$$a^{-3}\frac{d(a^3 n_p)}{dt} = \sum_{i=1}^{\infty} \lambda_{rec}^{(i)}(T) \left[-n_p n_e + n_{H^{(1)}} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\chi_H/T} \right]$$

But there's a catch.

Ionizations and recombinations to ground state $^{(1)}$, through Lyman transitions, are so fast that they exceed rate at which UV radiation field can thermalize. So these transitions

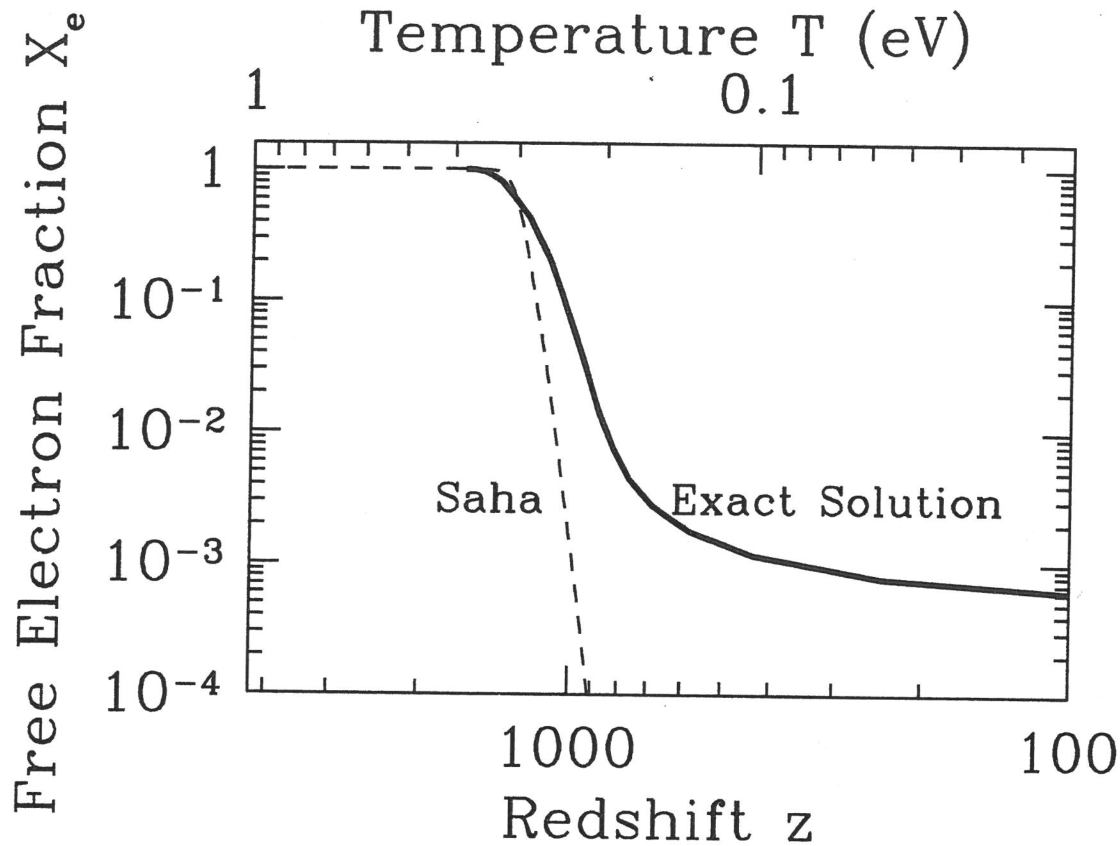


Figure 3.4. Free electron fraction as a function of redshift. Recombination takes place suddenly at $z \sim 1000$ corresponding to $T \sim 1/4$ eV. The Saha approximation, Eq. (3.37), holds in equilibrium and correctly identifies the redshift of recombination, but not the detailed evolution of X_e. Here $\Omega_b = 0.06, \Omega_m = 1, h = 0.5$.

The computation of the neutron/proton ratio affects the abundance of light elements today. Similarly, the evolution of the free electron abundance has major ramifications for observational cosmology. Recombination at $z_* \sim 1000$ is directly tied to the *decoupling* of photons from matter.[4] This decoupling, in turn, directly affects the pattern of anisotropies in the CMB that we observe today.

Decoupling occurs roughly when the rate for photons to Compton scatter off electrons becomes smaller than the expansion rate.[5] The scattering rate is

$$n_e \sigma_T = X_e n_b \sigma_T \tag{3.43}$$

where $\sigma_T = 0.665 \times 10^{-24}$ cm^2 is the Thomson cross section, and I continue to ignore helium, thereby assuming that the total number of hydrogen nuclei (free protons + hydrogen atoms) is equal to the total baryon number. Since the ratio of the

[4]Notice from Figure 1.4 that even though photons stop scattering off electrons at $z \sim 1000$, electrons do scatter many times off photons until much later. This is not a contradiction: there are many more photons than baryons. In any event, many cosmologists shy away from the word *decoupling* to describe what happens at $z \sim 1000$ for this reason.

balance, and do not contribute to any net recombination. This is called "case B" recombination.

Case A: $\lambda^{A}_{rec}(T) = \sum_{i=1}^{\infty} \lambda^{(i)}_{rec}(T)$ all recombinations

Case B: $\lambda^{B}_{rec}(T) = \sum_{i=2}^{\infty} \lambda^{(i)}_{rec}(T)$ recombination to levels ≥ 2 only

Charge neutrality $\Rightarrow$ $n_e = n_p$.

Define $X_e \equiv \frac{n_e}{n_p + n_H} = \frac{n_e}{n_b} = \frac{n_p}{n_b}$

Because $T \sim .3\,eV \ll \chi_H = 13.6\,eV$, almost all H is in ground state, $n_H \approx n_{H^{(1)}}$.

$$\boxed{\frac{dX_e}{dt} = \lambda^{B}_{rec}(T)\left[-X_e^2 n_b + (1-X_e)\left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\chi_H/T}\right]}$$

Photon decoupling

Electron scattering (Thomson scattering)

$$e + \gamma \leftrightarrow e + \gamma$$

has cross-section $\sigma_T = \frac{8\pi}{3}\left(\frac{e^2}{m_e c^2}\right)^2$ independent of frequency.

$$\frac{\text{Scattering rate}}{\text{Expansion rate}} = \frac{n_e \sigma_T c}{H}$$

$$\approx 100\, X_e \text{ at } a \approx 1000$$ Dodelson eq. (3.46).

$X_e \approx 10^{-3}$ at end of recombination, so photons decouple at that time, traveling freely without further scattering.

WIMP freeze-out

Consider a (massive) particle X that becomes non-relativistic, but whose annihilations freeze out

$$X + \bar{X} \longleftrightarrow Y + Y$$

$$a^{-3}\frac{d(a^3 n_x)}{dt} = -n_x n_{\bar{x}}\,\lambda_{\rightarrow}(T) + \lambda_{\leftarrow}(T)$$

$$\frac{\lambda_{\leftarrow}(T)}{\lambda_{\rightarrow}(T)} = n_x n_{\bar{x}}\Big|_{TE}$$

$$= g_x^2\left(\frac{m_x T}{2\pi}\right)^3 e^{-2m_x/T} \qquad T \ll m_x$$

Define

$$Y \equiv \frac{n_x}{T^3}\,, \qquad a \equiv \frac{m_x}{T}$$

then

$$\frac{a\,dY}{da} = \frac{m_x^3\,\lambda_{\rightarrow}}{a\,H(m_x)}\Big[-Y^2 + \underbrace{g_x^2\left(\frac{a}{2\pi}\right)^3 e^{-2a}}_{\text{tiny near end of annihilation}}\Big]$$

Final value Y_∞ of Y satisfies

$$\frac{m_x^3\,\lambda_{\rightarrow}}{a_f\,H(m_x)}\,Y_\infty \approx 1$$

$$\text{ie}\quad \lambda_{\rightarrow} \approx \underbrace{\overbrace{(m_x/T_f)}^{\sim 10}\,\overbrace{[m_x^{-2}H(m_x)]}^{\text{independent of } m_x}}_{\underbrace{m_x Y_\infty}_{\propto\,\Omega_x \text{ independent of } m_x}}$$

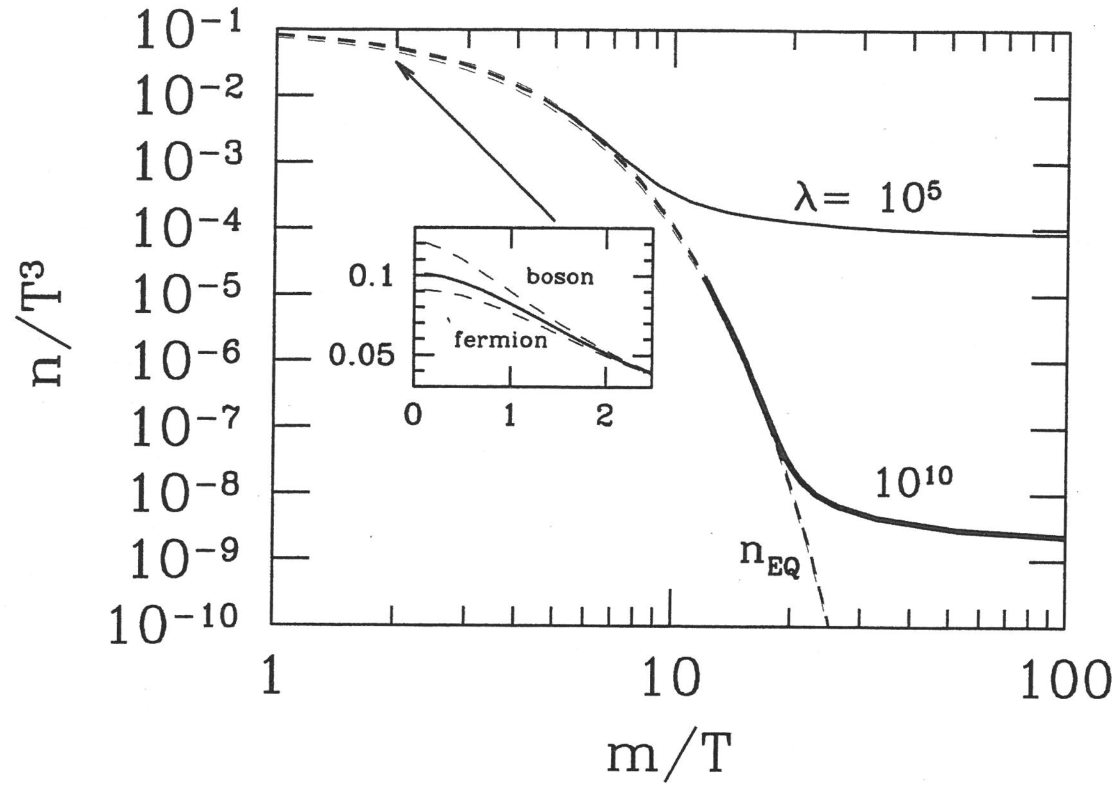


Figure 3.5. Abundance of heavy stable particle as the temperature drops beneath its mass. Dashed line is equilibrium abundance. Two different solid curves show heavy particle abundance for two different values of λ, the ratio of the annihilation rate to the Hubble rate. Inset shows that the difference between quantum statistics and Boltzmann statistics is important only at temperatures larger than the mass.

simply falls off as a^{-3}. So its energy density today is equal to $m(a_1/a_0)^3$ times its number density where a_1 corresponds to a time sufficiently late that Y has reached its aymptotic value, Y_∞. The number density at that time is $Y_\infty T_1^3$, so

$$\rho_X = mY_\infty T_0^3 \left(\frac{a_1 T_1}{a_0 T_0}\right)^3 \simeq \frac{mY_\infty T_0^3}{30}. \tag{3.57}$$

The second equality here is nontrivial. You might expect that aT remains constant through the evolution of the universe, so that the ratio a_1T_1/a_0T_0 would be unity. It is not, for the same reason that the CMB and neutrinos have different temperatures. We saw in Chapter 2 that photons are heated by $e^\pm$ annihilation, while neutrinos which have already decoupled are not. Similarly, as the universe expands, photons are heated by the annihilation of the zoo of particles with masses between 1 MeV and 100 GeV, so T does not fall simply as a^{-1}. You can show in Exercise 11 that as a result $(a_1T_1/a_0T_0)^3 \simeq 1/30$. Finally, to find the fraction of critical density today contributed by X, insert our expression for Y_∞ and divide by $\rho_{\rm cr}$:

$$\Omega_X = \frac{x_f}{\lambda}\frac{mT_0^3}{30}$$

Numerically

$$\lambda_{\rightarrow} = \frac{\pi}{30^{1/2}} \left(\frac{8\pi G}{3c\hbar}\right)^{3/2} \frac{g_{\circledast}}{g_*^{1/2}} \frac{(kT_0)^3}{c^4} \frac{a_f}{\Omega_x H_0^2}$$

$$= \frac{0.331 \times 10^{-41}\ m^2}{\Omega_x h} \left(\frac{a_f}{10}\right) \left(\frac{g_*}{100}\right)^{-\frac{1}{2}}$$

$\approx$ weak interaction cross-section.

Coincidence suggests that the non-baryonic Cold Dark Matter might be a WIMP

Particle must also be neutral, and stable.

Leading contender: LSP Lightest Supersymmetric Partner.