1st order ordinary diff eqs   RHB Ch 14

Q: What is a 1st order ODE?

A: $f\left(\underset{\{2,\frac{1}{2}\}}{x}, y, \frac{dy}{dx}\right) = 0$

Is $(y') + y = 0$ 1st order?

Q: What is fastest way to solve 1st order diff eq?

A: Mathematica. But does not find all solns. It will not necessarily find a nice substitution — you may well do better.

Run ch14.nb

Methods to solve 1st order ODEs

1. Separable eqs. Importance? Like, duh.

$$\frac{dy}{dx} = f(x)\,g(y)$$

Soln? You really should know that.

$$\frac{dy}{g(y)} = f(x)\,dx.$$

1'. Separable after substitution

(a) One important example is the dimensionally consistent equation:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

(RHB call this "homogeneous" — poor nomenclature)

Substitute dimensionless variable

$$v = y/x$$

$$\Rightarrow \frac{dy}{dx} = \frac{d(vx)}{dx} = x\frac{dv}{dx} + v = f(v) \Rightarrow \frac{dv}{dx} = \frac{f(v) - v}{x}$$

(b) A variation of this is

$$\frac{d\ln y}{d\ln x} = f\left(\frac{y^m}{x}\right)$$

(RHB call this "isobaric" — huh?)

Again substitute dimensionless variable

$$v = \frac{y^m}{x}$$

to get separable eq.

Moral: Apply dimensional analysis to your equation! It could simplify your life.

(c) RHB give further examples:

$$\frac{dy}{dx} = f(ax + by + c) \qquad a, b, c \text{ constant}$$

separable after substitution $z = ax + by + c$,

(d) $$\frac{dy}{dx} = \frac{ax + by + c}{ex + fy + g}$$

Separable after $x = X + \alpha$, $y = Y + \beta$

then $v = \dfrac{Y}{X}$

Like integration, there's no general method to recognize a good substition.

It's pattern recognition and guesswork.

Importance? ★

2. Exact eqs

Suppose you can find by inspection $f(x,y)$ such that

$$\frac{dy}{dx} = -\frac{\partial f/\partial x}{\partial f/\partial y}$$

This rearranges to

$$\underbrace{\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy}_{} = 0$$

But $= df$ = total derivative of f.

Solution of $df = 0$

is just $f(x,y) = $ constant.

Suppose you have

$$\frac{dy}{dx} = -\frac{a(x,y)}{b(x,y)}$$

one of those thms that's useful to remember

$$\boxed{\text{Frobenius' theorem: } a = \frac{\partial f}{\partial x} \text{ and } b = \frac{\partial f}{\partial y} \iff \frac{\partial a}{\partial y} = \frac{\partial b}{\partial x}}$$

Proof: $\Leftarrow$ is immediate. Q: why?

$\Rightarrow$ is tricky.

Given $a(x,y)\,dx + b(x,y)\,dy$

in principle

it is always possible to find integrating factor $\mu(x,y)$ such that

$$\mu a\,dx + \mu b\,dy = df = \text{total derivative},$$

but there is no general technique to find $\mu(x,y)$.

Importance? ★★★

3. Linear eqs

$$\frac{dy}{dx} + a(x)\, y = f(x)$$

RHB solve by finding integrating factor. However, it is more didactic to follow the general method of $\boxed{\text{Green's functions}}$. This is the simplest example of GF method.

Qualitatively..

Eq can be written

$$\mathcal{L} y = f$$

where $\mathcal{L}$ is linear differential operator

$$\mathcal{L} \equiv \frac{d}{dx} + a(x)$$

Q: In what sense is $\mathcal{L}$ linear?

A: $\mathcal{L}(y_1 + y_2) = \mathcal{L}y_1 + \mathcal{L}y_2$

If $\mathcal{L}$ were a matrix, soln would be

$$y = \mathcal{L}^{-1} f$$

Here $\mathcal{L}$ is a differential operator, and its inverse

$\mathcal{L}^{-1}$ is an integral operator.

The integrand of the integral operator $\mathcal{L}^{-1}$ is the Green's function of $\mathcal{L}$.

I think of GF as being $\mathcal{L}^{-1}$ itself.

GF's are also called "propagators" in QFT.

$$\boxed{\text{A GF is the inverse } \mathcal{L}^{-1} \text{ of a linear diff op } \mathcal{L}}$$

If $\mathcal{L}$ were a matrix, to get its inverse you'd solve

$$\mathcal{L}_{ij} \mathcal{L}^{-1}_{jk} = \underset{\substack{\text{unit}\\\text{matrix}}}{1_{ik}}.$$

But in fact $\mathcal{L}$ is a differential op

Seek solution to

$$\mathcal{L} \underbrace{G(x, x_0)}_{\substack{\equiv \text{GF, equiv}\\ \text{to } \mathcal{L}^{-1}}} = \underbrace{\delta(x - x_0)}_{\substack{\text{equiv to}\\ \text{unit matrix.}}}$$

Q: Why is $\delta(x - x_0)$ the unit matrix?

A: Because $\int \delta(x - x_0)\, f(x_0)\, dx_0 = f(x)$

$$\sum_j 1_{ij} \quad f_j \qquad = f_i$$

⑥ $\Longrightarrow$

Need to solve

$$\left(\frac{d}{dx} + a(x)\right) G(x, x_0) = \delta(x - x_0) = 0 \text{ except at } x = x_0$$

Homog eqn is

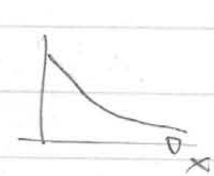
$$\left(\frac{d}{dx} + a(x)\right) y_{hom}(x) = 0$$

Homog soln is

$$\frac{dy_{hom}}{dx} = -a(x)\, y_{hom}$$

ie $\dfrac{dy_{hom}}{y_{hom}} = d\ln y_{hom} = -a(x)\, dx$

ie $y_{hom}(x) = \text{constant} \, . \, \exp\left(-\int^x a\, dx\right)$



Thus $G(x, x_0) = \begin{cases} 0 & x < x_0 \\ \text{const} \, . \, y_{hom} & x > x_0 \end{cases}$

Physically, GF represents the "response" to a delta-function "impulse" at $x = x_0$.
Prescription for GF requires boundary conditions, as you might expect for an integral operator.

Bcs depend on physical problem.
If x is time, then might expect impulse to affect future.
Thus expect GF

$$G(x, x_0) \begin{cases} = 0 & \text{if } x < x_0 \text{ (past)} \\ \neq 0 & \text{if } x > x_0 \text{ (future)} \end{cases}$$

$\Leftarrow$ Back to 5.

At $x = x_0$, RHS is δ-fn.

Take integral over tiny interval about $x = x_0$

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \left(\frac{dG}{dx} + aG\right) dx = \underbrace{\int_{x_0-\varepsilon}^{x_0+\varepsilon} \delta(x-x_0)\, dx}_{=1}$$

$$\text{LHS} = \int_{x_0-}^{x_0+} \frac{dG}{dx}\, dx + \int_{x_0-}^{x_0+} aG\, dx$$

$$= \left[G\right]_{x_0-}^{x_0+} + 0$$

$$= G(x_0+) - G(x_0-) \xrightarrow{} 0$$

So
$$\boxed{G(x, x_0) = \begin{cases} 0 & x \le x_0 \\ \exp\left(-\int_{x_0}^{x} a\, dx\right) & x > x_0 \end{cases}}$$

Particular soln is

$$y_{part}(x) = \int G(x, x_0) f(x_0)\, dx_0$$

$$= \int^{x} \exp\left(-\int_{x_0}^{x} a\, dx\right) f(x_0)\, dx_0$$

This is $\mathcal{L}^{-1} f$ and you see that $\mathcal{L}^{-1}$ is an integral operator as claimed.

Gen soln to $\mathcal{L}y = f$ is

$$y(x) = \underset{\substack{\uparrow \\ \text{some constant.}}}{C}\, y_{hom} + y_{part}$$

Higher-order ODEs

Q: How to solve numerically? n'th order diff eq

$$\frac{d^n y}{dx^n} + a_{n-1}(x,y)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x,y)\frac{dy}{dx} + a_0(x,y) = 0.$$

$y^{(n)}$, $y^{(n-1)}$, $y^{(1)}$

A: Break into n coupled 1st order diff eq

$$\frac{dy^{(m)}}{dx} = y^{(m+1)} \qquad m = 0, \dots, n-1$$

and

$$\frac{dy^{(n-1)}}{dx} = -a_{n-1}y^{(n-1)} - \dots - a_1 y^{(1)} - a_0 .$$

Q: How many boundary conditions?

A: n,

B.c.s may be:

(1) Initial value (hyperbolic):
$y^{(m)}$ $m = 0, \dots, n-1$ specified at some initial x.

(2) Boundary value (elliptic)
$y^{(m)}$ specified at 2 (or more) points x.

(1) (much) easier — eg. RK scheme

(2) harder - if you can rearrange diff eq into init value problem, do so.

Q: What is 2nd order ODE?
A: Eq of form
$$F(x, y, y', y'') = 0.$$

The simplest 2nd order ODE is:
<u>2nd order linear ODE with constant coeffs</u>
$$\frac{d^2y}{dx^2} + 2a\frac{dy}{dx} + by = f(x) \quad a, b \text{ constant.}$$
E.g. simple harmonic oscillator eqn.
$$y'' + \omega^2 y = 0$$
Eq is actually forced ($f(x) \neq 0$), damped ($a \neq 0$), SHO.
Linear ODEs/PDEs with constant coeffs occur whenever you do pert theory on uniform background. Very common problem.
Most commonly PDEs, but ODE is prototype.
Q: Examples?
A: • Waves of any kind — Sound, Water, Plasma, Electromagnetic, Gravitational (GR? Yes!), Quantum Field Theory, String theory.
• Growth of linear fluxs in cosmology.

Q: Why did I require uniform background?
A: So coeffs (eg a, b above) are indept of "position" x, ie they are constant.
Q: Why are coeffs a, b indept of y, for small perts?
A: Because if y is small, then y^2 (or higher powers of y) are negligible.
Q: Faced with linear ODE/PDE with constant coeffs, what do you do?
A: Fourier transform!

Q: What's special about Fourier transform

A: Fourier modes are eigenmodes of translation operator.

Deep, important result:

If background possesses a certain symmetry, then expand perturbations in eigenmodes of that symmetry.

Why? Because ODEs/PDEs are then transformed from diff eqs to algebraic eqs.

Q, If background has rotational symmetry,

eg. Any spherical object
- ~~Atom~~
- Planet
- Star
- CMB

then expand perts in eigenfns of that symmetry.

What is rot operator?

A: Ang mom operator $\vec{L}$

Q: What are eigenmodes of $\vec{L}$?

A: Spherical harmonics.

Well, we're getting ahead of today's topic. Today we'll solve linear 2nd order ODE without using FT, and you'll see that Fourier modes - or at least exponentials - do in fact emerge.

determine evolution of pert

perturbation

"forcing" function:
an outside influence specified as fn of time (?) x.

Back to

2nd order linear ODE with const coeffs

$$\mathcal{L} y = f(x)$$

where $\mathcal{L}$ is linear diff operator

$$\mathcal{L} \equiv \frac{d^2}{dx^2} + 2a \frac{d}{dx} + b \qquad a, b = \text{const.}$$

Q: You meet this eqn in comps 1
(actually they'll probably give you a PDE, to make it a bit more difficult).
What to do?

A (a) FT! ← general strategy
(b) Try $y = e^{\lambda x}$ ← probably simpler to solve in comps panic situation.

2 parts to solution:

(1) Solve homog eqn

$$\mathcal{L} y_{hom} = 0$$

For n'th order eqn this will have n linearly indepr solns y_i

Then $y_{hom} = A_1 y_1 + \dots + A_n y_n$, A_i constants.

Q: Is each of these y_i a soln of homog eqn?

A: Yes.

(2) Solve particular eqn

$$\mathcal{L} y_{part} = f$$

Then general soln to $\mathcal{L} y = f$ is

$$y = y_{part} + A_1 y_1 + \dots + A_n y_n$$

Q: How to find y_{part}?

A: Use Green's function.

Homog soln

Plug $y = e^{\lambda x}$ into

$$y'' + 2a y' + by = 0$$

$$\Rightarrow (\lambda^2 + 2a\lambda + b) e^{\lambda x} = 0.$$

Solns are

$$\lambda_\pm = -a \pm \sqrt{a^2 - b}$$

So

$$y_{hom} = A_+ e^{\lambda_+ x} + A_- e^{\lambda_- x}.$$

Cases:

(a) $a^2 > b$: λ both real.

Depending on signs, solns may be growing or decaying or one of each.

(b) $a^2 < b$:

Let $q \equiv \sqrt{b - a^2}$ is real, $b = a^2 + q^2$

Then

$$\lambda_\pm = -a \pm iq$$

so

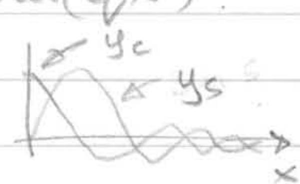
$$y_{hom} = A_+ e^{(-a+iq)x} + A_- e^{(-a-iqx)},$$

Equivalently

The case we'll look at to find ypart.

$$y_{hom} = A_c e^{-ax} \cos(qx) + A_s e^{-ax} \sin(qx)$$

If $a > 0$, solns are decaying and oscillating.



Physically coeff a represents kind of "friction", causing decay.

(c) $a^2 = b$.

Then trial $y = e^{\lambda x}$ gives only one soln $y = A e^{-ax}$.

Q: Oops, shouldn't there be two solns?

A: Yes,

Q: What is the general method to get a 2nd solution?
A: Wronskian.
Later...

Particular soln

$\mathcal{L}y = f$.

GF is soln of

$\mathcal{L}\,G(x,x_0) = \delta(x - x_0) \quad = 0 \quad$ for $x \neq x_0$

so

$$G(x,x_0) = \begin{cases} \text{a hom soln} & x < x_0 \\ \text{another hom soln} & x > x_0. \end{cases}$$

GF requires b.c.s. Q: how many b.c.s for 2nd order
A: 2.

What these are depends on the problem, ~~but~~ physically most likely here ~~case~~ is to set

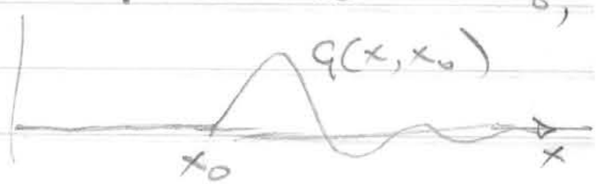
$G = \dfrac{dG}{dx} = 0$ at $x = x_0 - \varepsilon$

For nth order eqs, all derivs up to $G^{(n-2)}$ must be cts.

so $G(x,x_0) = 0$ for $x < x_0$.

we'll see ~~and~~ G must be cts at $x = x_0$ in order for G' to be well-def

Corresponds to physical situation where $\delta(x - x_0)$ is an impulse at $x = x_0$, which then evolves:



$G = \int \frac{dG}{dx}\,dx$

To solve for behavior at $x = x_0$, integrate over tiny interval about $x = x_0$

$$\int_{x=x_0-\varepsilon}^{x_0+\varepsilon} \left(\frac{d^2}{dx^2} + 2a\frac{d}{dx} + b\right) G(x,x_0)\,dx = \int_{x=x_0-\varepsilon}^{x_0+\varepsilon} \delta(x - x_0)\,dx$$

ie $\left[\frac{dG}{dx} + 2aG \right]_{x_0-\varepsilon}^{x_0+\varepsilon} = 1$

$[G]_{x_0-\varepsilon}^{x_0+\varepsilon} = 0 \qquad G = \int \frac{dG}{dx}\, dx$

so

$\left[\frac{dG}{dx} \right]_-^+ = 1$.

Q: For osc hom, what is soln that $= 0$ at $x = x_0$?

A: $G = A e^{-a(x-x_0)} \sin[q(x-x_0)]$

for sin

$\left.\frac{dG}{dx}\right|_{x=x_0} = A\{-a\, e^{-a(x-x_0)} \sin[q(x-x_0)] + q\, e^{-a(x-x_0)} \cos[q(x-x_0)]\}$

$= Aq$.

So $= 1$

$\Rightarrow A = \frac{1}{q}$

So $G(x, x_0) = \begin{cases} 0 & x < x_0 \\ \frac{1}{q} e^{-a(x-x_0)} \sin[q(x-x_0)] & x > x_0 \end{cases}$

So a particular soln to $\mathcal{L} y_{part} = f(x)$ is

$$\boxed{y_{part}(x) = \int G(x, x_0)\, f(x_0)\, dx_0}$$

~~x upper limit because $G(x,x_0) = 0$ for $x_0 > x$~~

$$= \int_{-\infty}^{x} \frac{1}{q} e^{-a(x-x_0)} \sin[q(x-x_0)]\, f(x_0)\, dx_0.$$

$$= \frac{1}{q} \mathrm{Re} \left[e^{-(a+iq)x} \int_{-\infty}^{x} e^{(a-iq)x_0} f(x_0)\, dx_0 \right]$$