ASTR 2030 Black Holes Spring 2019. Homework 3. Due in class Wed Apr 17.

Your name and ID:

Please write your answers on these sheets, and **show your working**!

Black Hole Threatens Earth

An errant black hole strays into the solar system, and it's headed for Earth, causing world-wide panic. What will happen?

How does a scientist set about thinking about an outlandish problem like that of a black hole eating the Earth? There is no observational evidence to go on, and no way to do a real experiment (we hope!). At first sight the problem might seem horribly complex (and so it is, in its full glory). However, is is possible to make progress by trying to identify what are the most important essentials of the problem, and simplifying the problem to the point where it becomes tractable. Oftentimes one has to guess what might be the most important effects, and later on check that those guesses are self-consistent.

The main trick that we will use is one that to my knowledge was first used by Sir Arthur Eddington to estimate the temperature at the center of the Sun. The idea is that, in any system where gravity is balanced by the motion of particles, the characteristic velocity v of particles will be approximately equal to the gravitational escape speed $(2GM/r)^{1/2}$

$$v = \left(\frac{2GM}{r} \right)^{1/2} . \quad (1)$$

In the case of a black hole plunging through the Earth, we interpret this equation as meaning that particles at distance r from the black hole, of mass M, will have characteristic speeds v. Rearranging equation (1) for the distance r at a given characteristic velocity v gives

$$r = \frac{2GM}{v^2} . \quad (2)$$

At the outset, we do not necessarily know what this 'characteristic speed' v refers to, specifically. Maybe it is the characteristic infall velocity of particles falling into the black hole. Or maybe it is the characteristic thermal velocity of particles at high temperature. Or maybe it is the characteristic velocity of electrons in an electron-degenerate gas. Let us proceed, and see what happens.

(a) Schwarzschild radius

The Schwarzschild radius r_s occurs where the particle velocity v equals the speed of light c

$$r_s = \frac{2GM}{c^2} \ . \tag{3}$$

Evaluate the Schwarzschild radius r_s for a black hole whose mass happens to equal the mass $M = M_\oplus = 6 \times 10^{24}\,\mathrm{kg}$ of the Earth. [Constants: $G = 6.7 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$; $c = 3 \times 10^8\,\mathrm{m\ s^{-1}}$.]

For $M = M_\oplus$, the Schwarzschild radius r_s is ______________________ meters.

(b) Distance versus velocity

Equations (2) and (3) show that the distance r from the black hole in units of the Schwarzschild radius r_s is related to the characteristic velocity v in units of the speed of light c by

$$\frac{r}{r_s} = \left(\frac{v}{c}\right)^{-2} \ . \tag{4}$$

Plot this relation as a **straight line** on the graph at the end of these sheets. [Hint: Try some values of v/c, such as $v/c = 1$, $v/c = 10^{-2}$, and $v/c = 10^{-4}$.]

(c) Distance at Earth escape velocity

The escape velocity from the surface of Earth is about $v = 11{,}200\,\mathrm{m\ s^{-1}}$. What fraction of the speed of light c is this? **Label** this point on the straight line that you drew in part (b), and read off from the graph the distance r from the black hole in Schwarzschild radii.

The velocity v is ______________________ c.

The distance r is ______________________ r_s.

For the particular case of a black hole with mass equal to an earth mass, $M = M_\oplus$, use your result from part (a) to translate this distance r into kilometers. This distance should equal the radius of the Earth.

For $M = M_\oplus$, the distance r is ______________________ km.

(d) Degeneracy distance

Far from the black hole, Earth rock will be either solid or molten, and as such will be almost incompressible. But close enough to the black hole, the pressure will be high enough that the rock will transform into a compressible electron degenerate vapor. This happens when electron velocities exceed the atomic velocity of about $v \approx \frac{1}{137}c$. **Label** this point on the straight line that you drew in part (b), and read off from the graph the distance from the black hole in Schwarzschild radii.

The degeneracy distance r is ____ r_s.

What is the velocity v in m s^{-1} at the transition point?

The velocity v at the degeneracy point is ________________________________ m s^{-1}.

What is the radius r in m (meters) for the case where the black hole mass is an Earth mass, $M = M_\oplus$?

For $M = M_\oplus$, the degeneracy distance r is ________________________________ m.

(e) Mass infall rate

The mass per unit time $\dot{M}$ (the dot on top of $\dot{M}$ signifies a rate per unit time) at which matter at distance r falls into the black hole equals the area $4\pi r^2$ times the velocity v times the density ρ

$$\dot{M} = 4\pi r^2 v \rho \ . \tag{5}$$

Assume (why? — read the paragraph on the next page) that the appropriate distance r and velocity v to be used in this equation are those at the degeneracy transition point from part (d), and assume that the density is an Earth density $\rho = 5{,}000\,\mathrm{kg\,m^{-3}}$. Evaluate the mass infall rate $\dot{M}$ for the case of a black hole with mass equal to an earth mass, $M = M_\oplus$.

For $M = M_\oplus$, the mass infall rate $\dot{M}$ is ________________________________ kg s^{-1}.

Show that the mass infall rate (5) varies with black hole mass as

$$\dot{M} \propto M^2 \ . \tag{6}$$

[Hint: The $\propto$ sign means "is proportional to", which means "is equal to some constant multiplied by". The velocity v from part (d), and the Earth density ρ, are both fixed constants, right? So all you have to worry about is how the distance r from part (d) depends on the black hole mass M.]

[Why is it correct to evaluate the mass infall rate at the point where the rock transitions from an incompressible solid/liquid to a compressible electron degenerate vapor? If instead you supposed that the mass infall rate were given by equation (5) evaluated at a larger distance, then you would find that the mass infall rate would be larger at larger distance, going as $\dot{M} \propto r^{3/2}$ for incompressible rock. But this cannot be true, because the mass infall rate should adjust itself to the same value at all distances. In other words, incompressible rock closer to the black hole must block incompressible rock further away, stopping it from falling in rapidly. The characteristic velocities of particles at large distances from the black hole must be interpreted not as infall velocities, but rather as random velocities, which act as a pressure that keeps incompressible rock from falling directly into the black hole.

On the other hand, rock vapor in the electron degenerate phase is quite compressible, and the density ρ is able to increase rapidly (as $r^{-3/2}$) near the black hole. The result is that rock vapor falls almost freely on to the black hole from the distance where the rock transitions from an incompressible solid/liquid into a compressible electron degenerate vapor.]

(f) Growth time

The characteristic timescale for the black hole to increase its mass is

$$t = \frac{M}{\dot{M}} \ . \tag{7}$$

Evaluate the timescale t in **years** (1 year $= 3 \times 10^7$ seconds) for the case of a black hole with mass equal to an earth mass, $M = M_\oplus$.

For $M = M_\oplus$, the growth timescale t is ______________________________ yr.

From equation (7) and the proportionality (6) deduce that the timescale varies inversely with black hole mass M

$$t \propto M^{-1} \ . \tag{8}$$

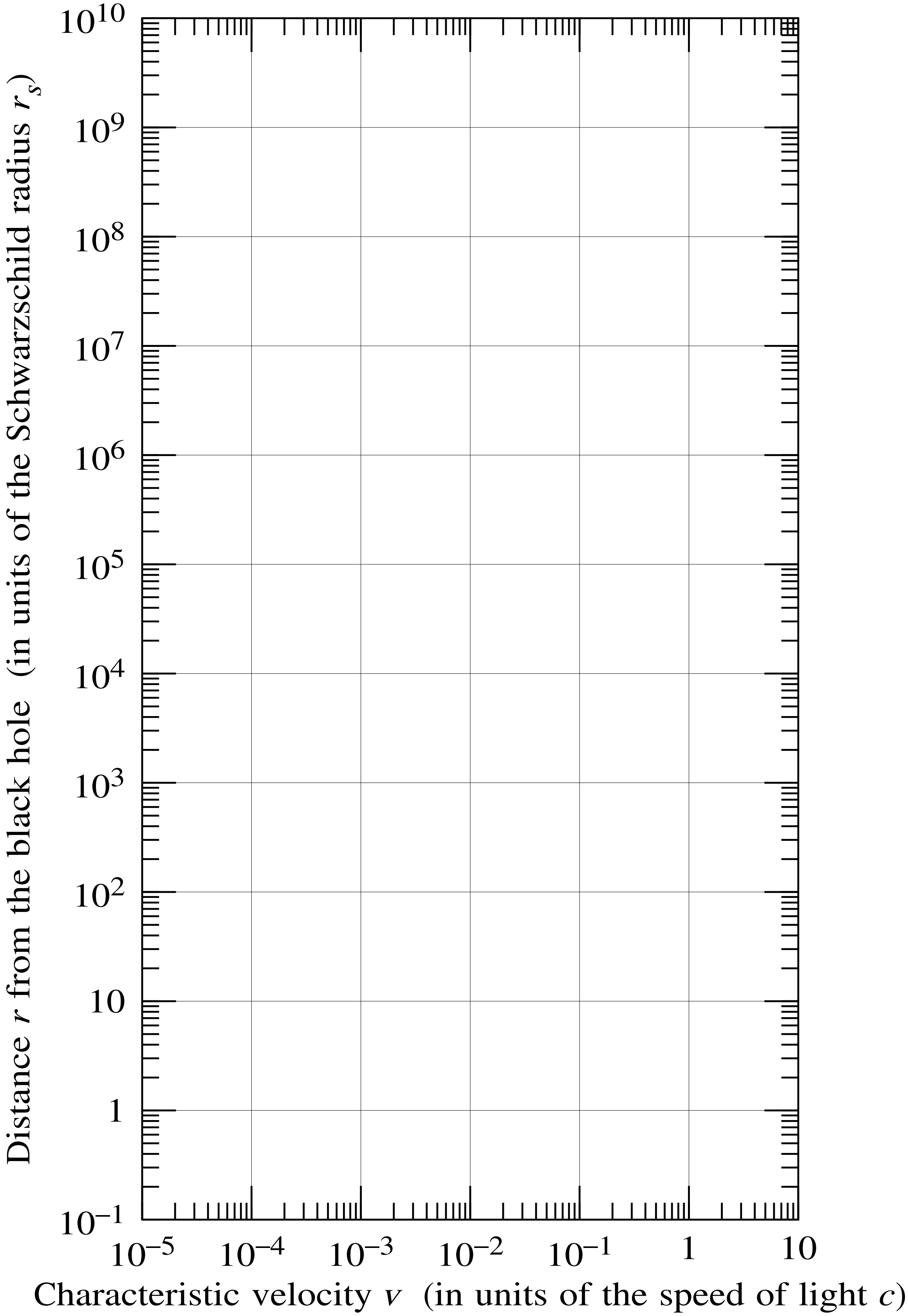
(g) Story (1% extra credit for the best answer)

You are contracted to write a Hollywood script involving a black hole that plunges through the Earth. What mass black hole would you choose, and why? [Hint: There is not necessarily a correct answer here. What I want you to do is to make practical sense of what you have discovered in this homework.]

(h) Postscript (no credit)

The above analysis is simplified. Among other things, it ignores the fact that a black hole arriving from elsewhere in the Milky Way is likely to hit the Earth at up to $300\,\mathrm{km\,s^{-1}}$, the escape velocity from the Milky Way. This is much faster than the speed of sound in the Earth, so the black hole should drive a shock wave in front of it, like a supersonic jet. However, I don't think the shock will change much the mass infall rate we estimated, because the $300\,\mathrm{km\,s^{-1}}$ is smaller than the atomic velocity $\frac{1}{137}c$. Rather than stopping inside the Earth, the black hole will drill its way through the Earth without hardly slowing down at all. It will come out the other side of the Earth, and continue on its merry way.

The analysis also ignores Hawking radiation, which can be important for a black hole more massive than a modest-sized asteroid.



10^10
10^9
10^8
10^7
10^6
10^5
10^4
10^3
10^2
10
1
10^−1
10^−5
10^−4
10^−3
10^−2
10^−1
1
10
Distance r from the black hole (in units of the Schwarzschild radius r_s)
Characteristic velocity v (in units of the speed of light c)