I. SPECIAL RELATIVITY

Physicist : describes
Philosopher : asks why?
People : do both

Structure of spacetime

(a) $\left.\begin{array}{ll}\text{Space} & \text{3D}\\ \text{Time} & \text{1D}\end{array}\right\}$ form a 4D manifold

a topological space which looks locally like $\mathbb{R}^4$

Why?
What if space were 1 D ?
7 D ?
What if time were 2 D ?

(b) Space & time intervals can be measured
space – rulers
time – clocks

How do clocks know to tick at a certain rate?

In everyday life, different observers obtain reproducible measurements of space & time. Why (not)?

(c) Does spacetime have an existence independent of objects (observers) in it?

- Yes – Galileo Galilei (1564 - 1642)
  " – Isaac Newton (1643 - 1727)
- No – Einstein

Do fundamental particles know about the structure of spacetime?
Apparently yes!
In quantum mechanics, particles have
- frequency ν $E = h\nu$
- wavelength λ $p = h/\lambda$
- spin ← so they act like as if they were gyroscopes: they know about direction in space.

Inertial spacetime frames

Newton's 1st law:
"Body moves in a straight line at constant velocity, unless acted on by forces."

A system of spacetime coordinates with respect to which unaccelerated bodies move in straight lines at constant velocity is called an $\boxed{\text{inertial frame}}$.

Can you think of examples of non-inertial frames?

- Globally inertial frame
  is inertial frame constructed over all spacetime. Existence thereof postulated by
    - Galileo/Newton
    - Special Relativity.

  Existence equivalent to assumption that spacetime has 4D Euclidean geometry.

  Abandoned by General Relativity, which postulates existence instead of
- Locally inertial frames,
  In a sufficiently small neighborhood of each spacetime point, there exists a coordinate system with respect to which unaccelerated bodies move in straight lines.

Symmetry

Arguably the single most important concept in all of modern physics.

Symmetry = laws of physics are unchanged (values of observable quantities are unchanged) when system is transformed in some way (e.g. translated, rotated, etc.).

Global symmetry

= Transform everything everywhere by the same amount.

Consequence (Noether's theorem):
corresponding to every global symmetry is a conservation law.

Local symmetry (applies to continuous, not discrete symmetries)

= Transform
- everything at same spacetime point by the same amount;
- things at different points by different amounts.

Consequence:
Forces!

Spacetime symmetries

Symmetry	Dimensions	Consened quantity
Translation in time	1D	Energy
Translation in space	3D	Momentum
Rotation in space	3D	Angular momentum
Velocity boost	3D	Velocity of center of mass
Time reversal $t \to -t$	Discrete	T
Spatial inversion $\mathbf{r} \to -\mathbf{r}$	"	P (Parity)

Other symmetries

Particle exchange	Discrete	system is bosonic ← integral ang. mom fermionic ← ½-integral ang. mom.
Particle ↔ antiparticle or "charge conjugation"	Discrete	C
Phase of wavefunction of charged particle	1D	Electric charge
⋮		

Can you think of a spacetime transformation which is not a good symmetry?

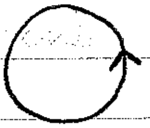
Symmetries & Forces

Symmetry	Force	"Strength"
Space-time symmetries	Gravity	$\sim 10^{-42}$
$U_{em}(1)$	Electromagnetism	$1/137$
$SU(2) \times U_Y(1)$	Electroweak	$\sim 10^{-4}$
$SU(3)$	Color	~ 1

Notes:

1. "Strength" means the relative ~~strength~~ of the force between two protons in a nucleus. A more precise definition can be made.
2. $\underbrace{U(1)}_{\text{unitary group of 1 dimension}}$, $\underbrace{SU(2)}_{\text{special unitary group of 2 dimensions}}$ etc denote certain compact groups.

   e.g. $U(1)$ is the group of rotations of a circle :



3. The electroweak symmetry $SU(2) \times U(1)$ appears only at energies $\gtrsim 100$ GeV (10^{15} K). At lower energies the electroweak symmetry is "spontaneously broken" into the electromagnetic and weak force. The strength of 10^{-4} is the strength of the broken weak force; at high energy, the strengths of the electromagnetic & weak forces become the same ($\sim 10^{-2}$).

Spatial 3-vectors

Vector = quantity with magnitude & direction.

3D vector notation is wonderful shorthand way to express & deal with spatial symmetries.

$\mathbf{r} = (x, y, z)$

vector — coordinates in one frame A

$\mathbf{r}' = (x', y', z')$

coordinates in another frame B

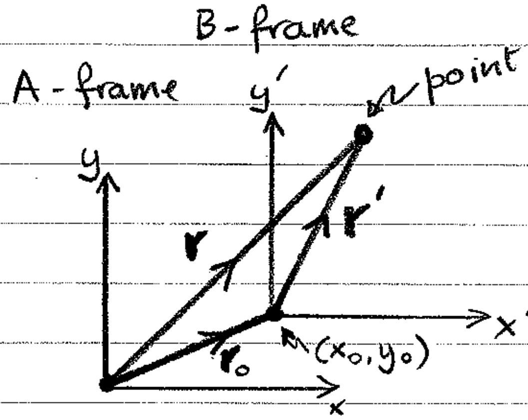
Vector notation recognizes that there is a certain arbitrariness (because of existence of symmetries!) in choice of coordinates.

Crucial point:

If $\mathbf{r} = \mathbf{s}$ in one frame

then $\mathbf{r}' = \mathbf{s}'$ in any other

Spatial translation



Frame B is translated by

$\mathbf{r}_0 = (x_0, y_0, z_0)$

with respect to A.

Then point

$\mathbf{r} = (x, y, z)$ in A frame

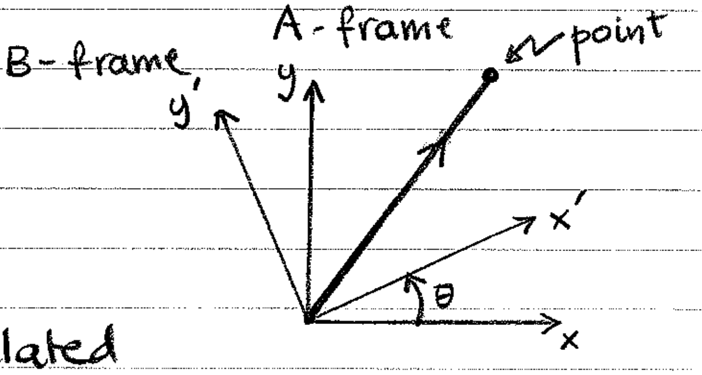
has coordinates

$\mathbf{r}' = (x', y', z')$

$= (x - x_0, y - y_0, z - z_0)$ in B frame

ie $\boxed{\mathbf{r}' = \mathbf{r} - \mathbf{r}_0}$

Spatial rotation



Frame B rotated by θ about z axis (for definiteness).

Then coordinates in B and A frames related by

$$x' = x\cos\theta + y\sin\theta$$
$$y' = -x\sin\theta + y\cos\theta$$
$$z' = z$$

which may also be written

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

or equivalently ← an "orthogonal" matrix, $\mathbf{O}^T = \mathbf{O}^{-1}$

$\boxed{\mathbf{r}' = \mathbf{O}\mathbf{r}}$

Galilean velocity transformation

Notice that Newton's 2nd law
a universal time exists

$$\mathbf{F} = m \frac{d^2\mathbf{r}}{dt^2}$$

is independent of both

- origin of coordinate system
- velocity of coordinate system

Crucial Galilean assumption: two systems in relative motion experience the same universal time.
We can assume (once synchronized)

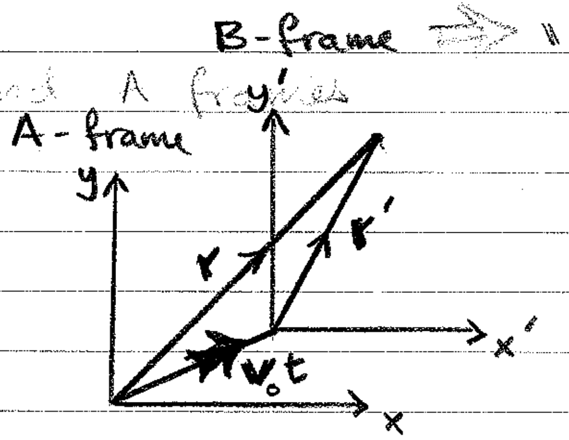
$$\boxed{t' = t}$$

If origin $\mathbf{r}_0$ of B relative to A frame moves at constant velocity $\mathbf{v}_0$, so

$$\mathbf{r}_0 = \mathbf{v}_0 t$$

Then coordinates in B & A frames are related by

$$\boxed{\mathbf{r}' = \mathbf{r} - \mathbf{v}_0 t}$$



Galilean law of velocity addition

Velocity of a point is

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} \qquad \text{in A system}$$

$$\mathbf{v}' = \frac{d\mathbf{r}'}{dt} \qquad \text{in B system}$$

ie $$\boxed{\mathbf{v}' = \mathbf{v} - \mathbf{v}_0}$$

Velocities add.

Crucial here that $\Delta t' = \Delta t$

Scalars

Key to scientific method is $\boxed{\text{reproducibility}}$.

Specifically, theories predict, and experiments / observations measure, the values of numbers.
Reproducible numbers – those whose values are independent of the coordinate system – are called $\boxed{\text{scalars}}$.

Galilean spacetime scalars

Galilean spacetime recognizes 2 classes of scalar:

1. Time intervals $\boxed{\Delta t}$ between two instants;
2. Spatial intervals (distances) Δr between two points

$$\boxed{\Delta r = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}}$$

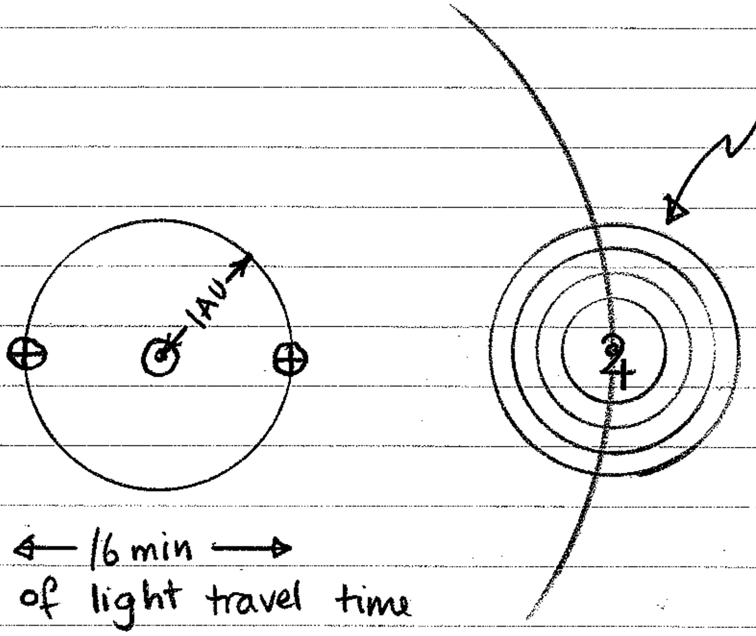
Δt and Δr are invariant under Galilean transformations

- translation of time
- " " space
- rotation of space

Light : the nemesis of classical Galilean spacetime – historical elements

- Light moves at a finite speed (1675).
- Light is a wave (1799).
  - erroneous inference : existence of an aether.
- Light is an electromagnetic wave (1864).
- Michelson & Morley failed to detect motion of $\oplus$ through aether (1887).
  - various theoretical ideas to explain this.
- Einstein's theory of Special Relativity (1905).

Olaus Roemer, Danish (1675), aged 33



Galileon satellites of Jupiter:

	Period (days)
Io	1.77
Europa	3.55
Ganymede	7.15
Callisto	16.69

Roemer noticed eclipses of Galilean satellites occurred 16 min later when ♃ was farthest from ⊕ than when ♃ was nearest ⊕.

$$\Rightarrow \text{light speed} \quad c = \frac{2\,\text{AU}}{16\ \text{min}}$$

Later, when AU (Astronomical Unit) measured,

$$\Rightarrow \quad c \approx 300{,}000\ \text{km/s}$$

Since 1983, define

$$\boxed{c \equiv 299{,}792{,}458\ \text{m/s}}$$

Dr. Thomas Young (1799), aged 26

demonstrated light is a wave by phenomena of
- interference – two slits
- diffraction – edge

in opposition to prevailing Newtonian corpuscular theory.

Q: Why is the wavelike nature of light difficult to see?

Aether – concept dates to Plato, Aristotle

1. What does it carry?
   - Light "luminiferous aether"
   - Forces - gravity
     - electrostatic force "many aethers"
     - magnetic force

2. What are its properties?
   - No viscosity (does not impede planets)
   - Pervades all space, penetrates vacuum / solids alike.
   - Aether waves have enormous velocity (= c) suggesting medium of tremendous rigidity.

From: M H Shamos "Great Experiments in Physics: Firsthand Accounts from Galileo to Einstein

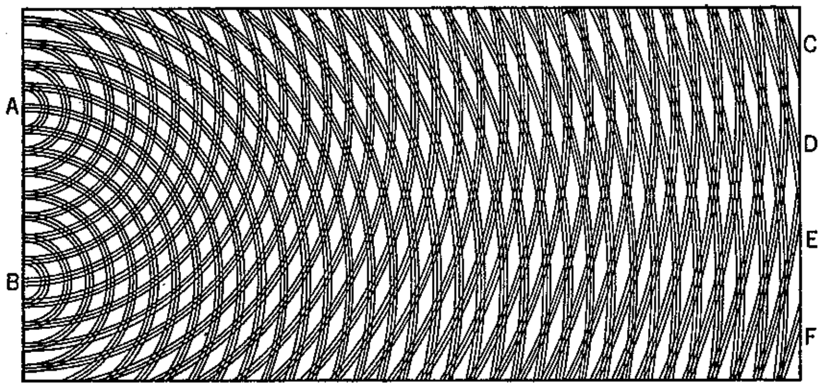


FIG. 1

and the bright stripes on each side are at such distances, that the light coming to them from one of the apertures, must have passed through a longer space than that which comes from the other, by an interval which is equal to the breadth of one, two, three, or more of the supposed undulations,■ while the intervening dark spaces correspond to a difference of half a supposed undulation, of one and a half, of two and a half, or more.■

That is, by an integral number of wavelengths.

An odd number of half wavelengths.

From a comparison of various experiments, it appears that the breadth of the undulations constituting the extreme red light must be supposed to be, in air, about one 36 thousandth of an inch, and those of the extreme violet about one 60 thousandth; the mean of the whole spectrum, with respect to the intensity of light, being about one 45 thousandth.■ From these dimensions it follows, calculating upon the known velocity of light, that almost 500 millions of millions of the slowest of such undulations must enter the eye in a single second. The combination of two portions of white or mixed light, when viewed at a great distance exhibits a few white and black stripes, corresponding to this interval; although, upon closer inspection, the distinct effects of an infinite number of stripes of different breadths appear to be compounded together, so as to produce a beautiful diversity of tints, passing by degrees into each other. The central whiteness is first changed to a yellowish, and then to a tawny color, succeeded by crimson, and by violet and blue, which together appear, when seen at a distance, as a dark stripe; after this a green light appears, and the dark space beyond it has a crimson hue; the subsequent lights are all more or less green, the dark spaces purple and reddish;■ and the red light appears so far to predominate in all these effects, that the red or purple stripes occupy nearly the same place in the mixed fringes as if their light were received separately.

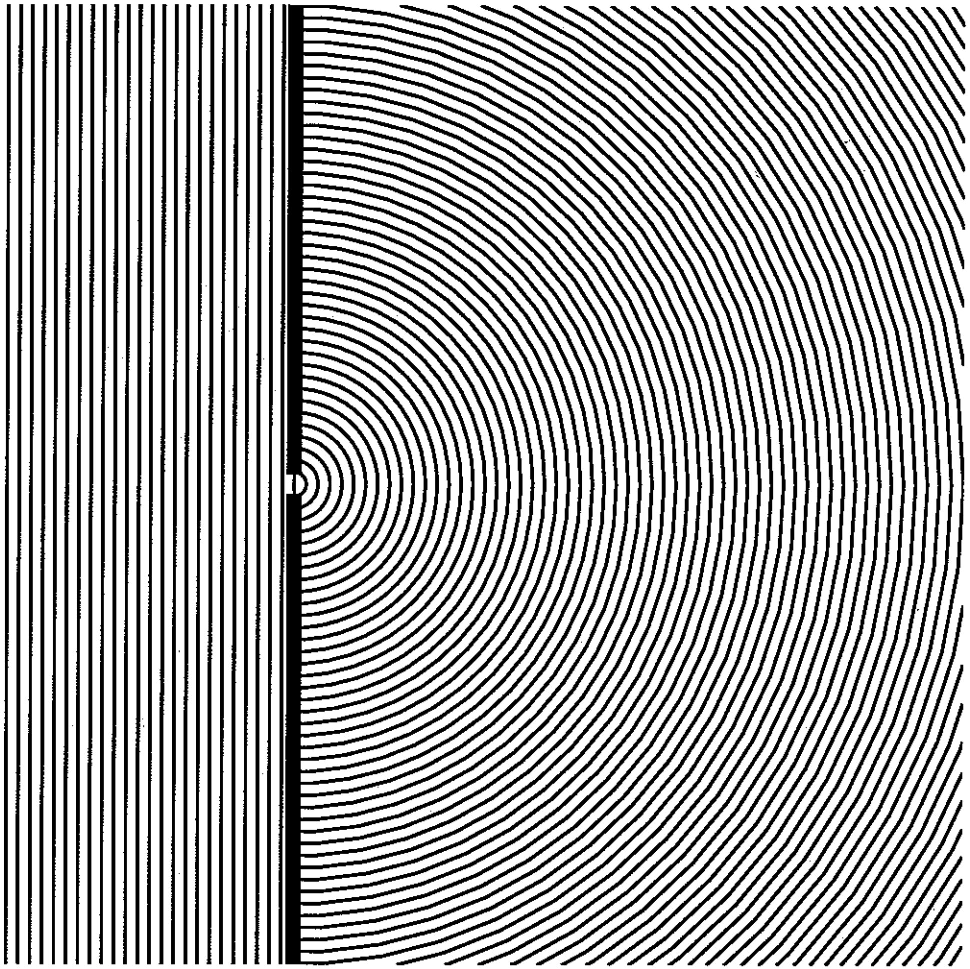
700 nm
420 nm

The actual range of visible light exceeds these limits only by small amounts. Young's observations were remarkably accurate.

The colors are formed as the different wavelengths satisfy the criterion for interference.

The comparison of the results of this theory with experiments fully establishes their general coincidence; it indicates, however, a slight cor-

1.16



James Clerk Maxwell, Scots (1831-1879)

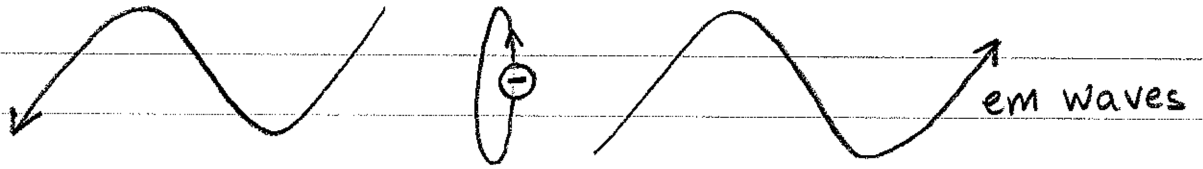
Unification of electricity, magnetism, light (1864).
Reduced 4 "aethers" to 2 (em & gravity).
Maxwell still believed in aether.

1. Forces of
   - electricity
   - magnetism

   unified into single $\boxed{\text{electromagnetic}}$ force

   Stationary electric charge produces electric field.
   Moving " " " magnetic " .

2. $\boxed{\text{Maxwell's equations}}$ looked like wave equations: wiggle charge, and it would produce an electromagnetic wave moving outward



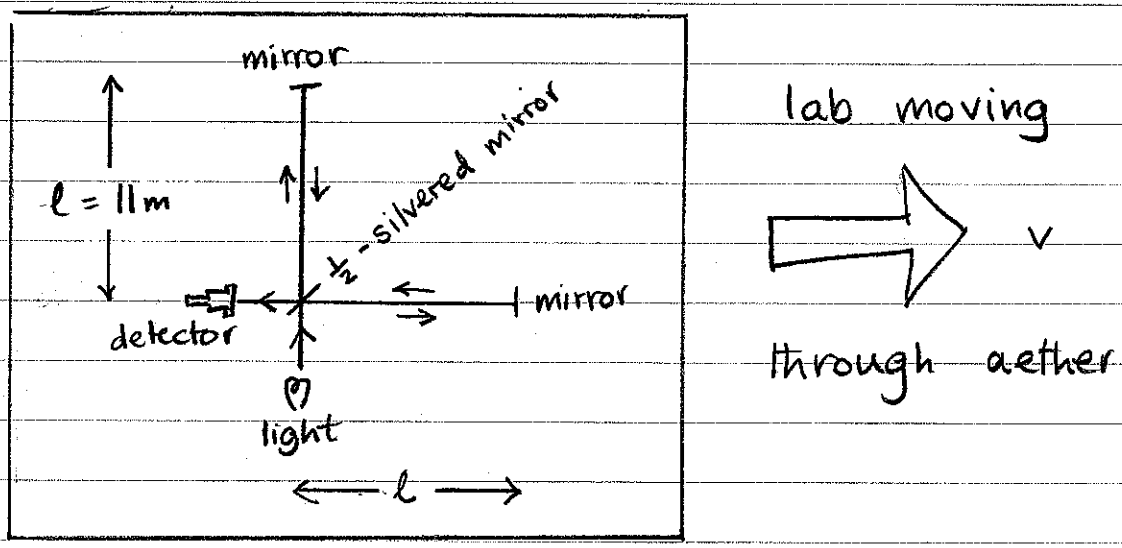
Remarkable thing was, the velocity of the waves predicted by the equations was numerically equal to known velocity of light.
Maxwell proposed

$$\boxed{\text{light = electromagnetic waves}}$$

Albert Michelson (1852 - 1931) + Edward Morley
Americans 1887 experiment

to detect motion of ⊕ through aether
as ⊕ travels around ☉.

Michelson interferometer (schematic)



Classically expect $\dfrac{\text{Time} \updownarrow}{\text{Time} \leftrightarrow} = \sqrt{1 - \frac{v^2}{c^2}}$

Problems:

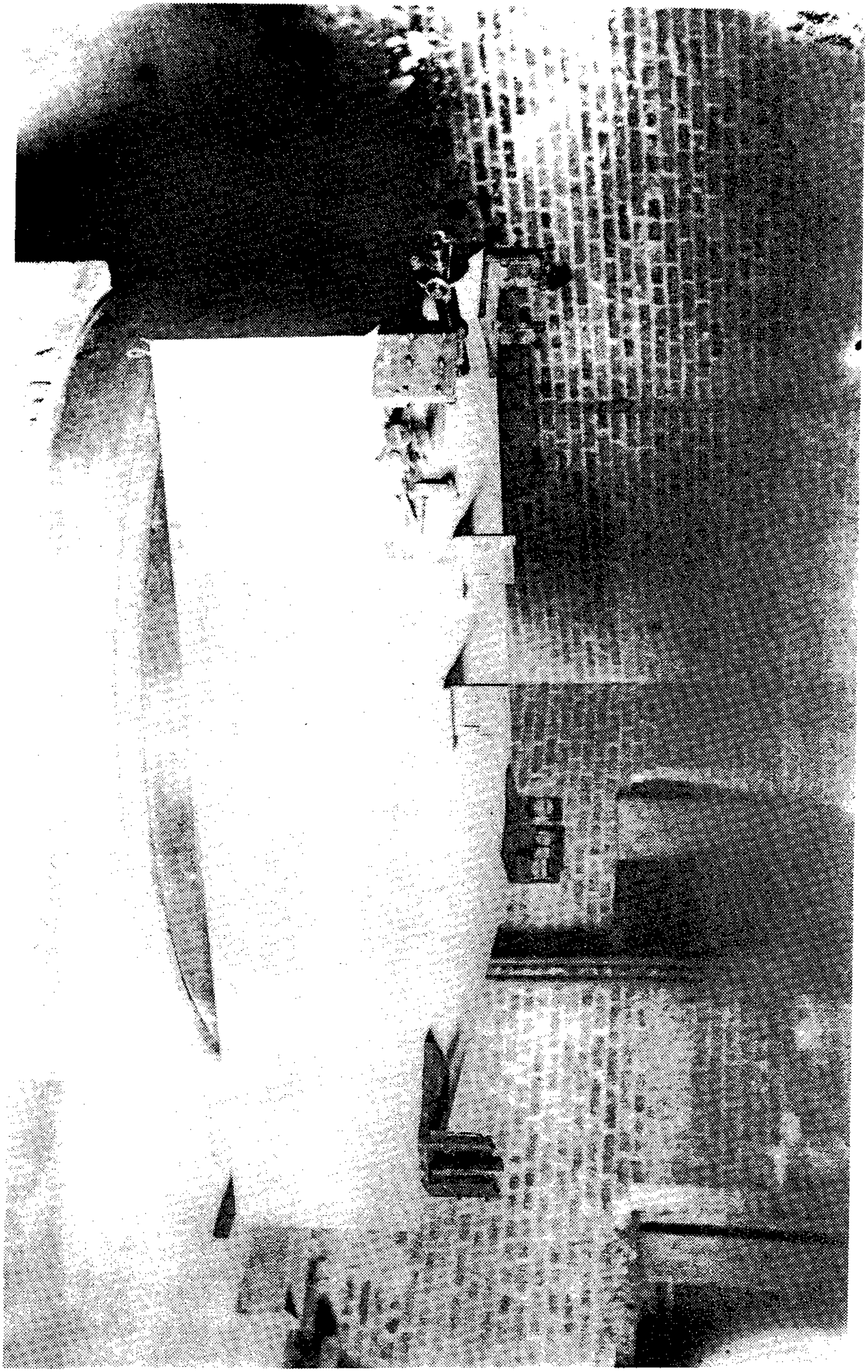
1. $V_\oplus \approx 30 \text{ km/s}$
$c \approx 300{,}000 \text{ km/s}$
$\Rightarrow \dfrac{V_\oplus}{c} \approx 10^{-4} \quad \Rightarrow \dfrac{V_\oplus^2}{c^2} \approx 10^{-8}$
$\Rightarrow$ difference in light travel time $\approx 10^{-8}$

2. Can't measure distance ℓ accurately.

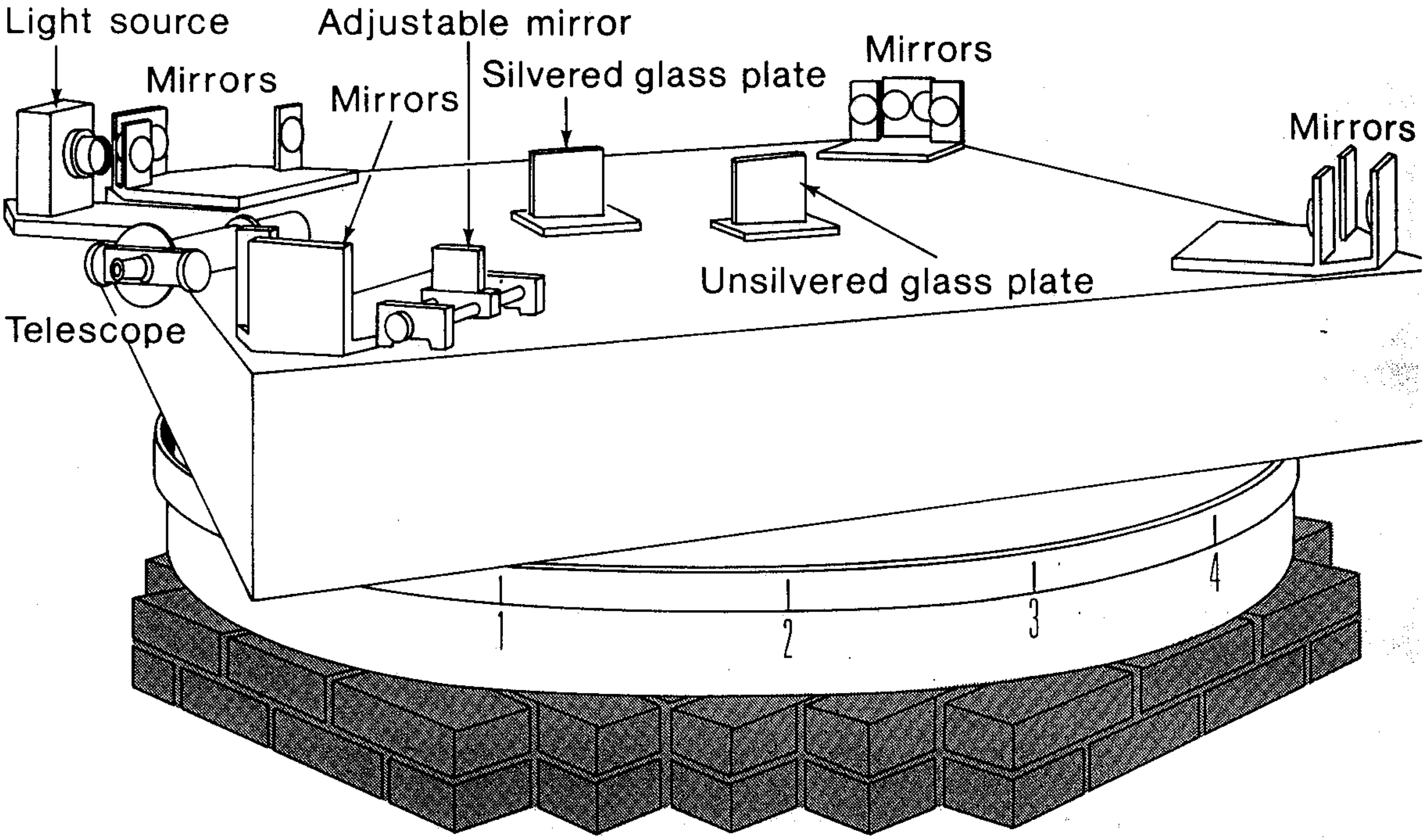
3. Can't guarantee $\ell_\updownarrow = \ell_\leftrightarrow$.

From: D.M. Livingston "The Master of Light:
A Biography of Albert A. Michelson"

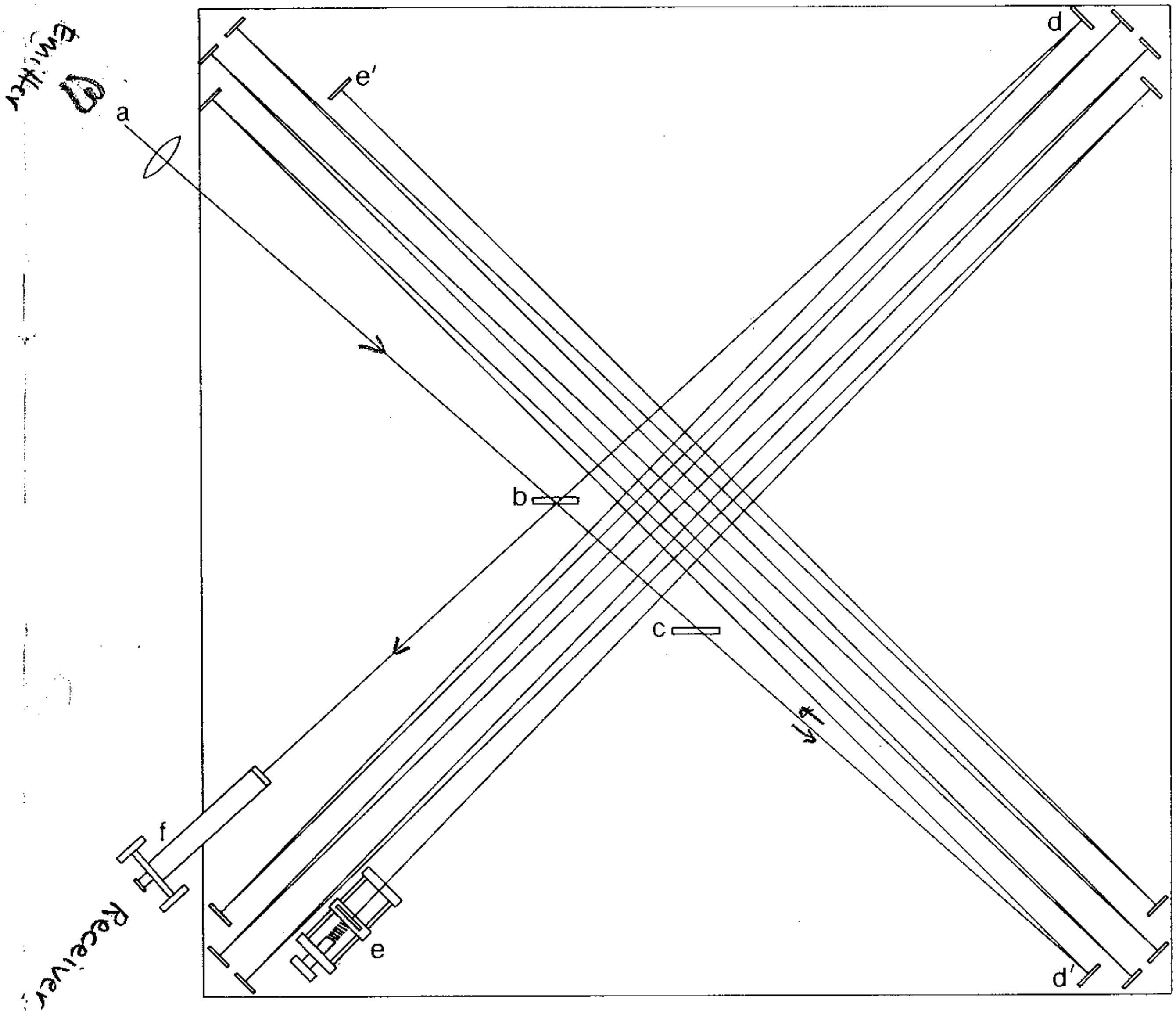
1.19
7



The interferential refractometer of the Michelson-Morley experiment (COURTESY HALE OBSERVATORIES)



The Cleveland interferometer of 1887, mounted on a stone floating in mercury



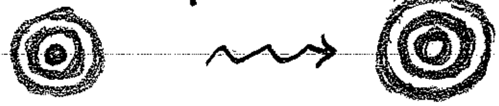
The paths of light in the 1887 interferometer

By using sixteen mirrors, instead of two as in the Potsdam experiment (see diagram, page 79), the light path was extended to 36 feet, thus increasing the sensitivity of the instrument tenfold. Light from *a* is split into two parts by the lightly silvered mirror at *b*. One ray is reflected to the plane mirror at *d* and is flashed back and forth between the four mirrors at *e* and at *d*. Retracing its path from *e* to *d*, it returns to *b* and is transmitted to *f*, the telescope. The second ray penetrates the mirror at *b*, is transmitted through *c* to *d′*, and is reflected between the mirrors at *d′* and those at *e′*, returning through *c* to *b*, where the interference may be examined as both rays enter the telescope at *f*.

(Diagrams on pages 127–129 adapted from Michelson and Morley, "On the Relative Motion of the Earth and the Luminiferous Ether," 1887)

Michelson - Morley solution

- Recombine beams to form interference pattern
- Rotate entire apparatus, watch for change in pattern



Change gives twice the difference in the path lengths ↕ vs. ↔, measured in wavelengths.

They choose $l \approx 11\,m \approx 10^8$ wavelengths of visible light so expected change by ≈ 1 fringe as rotated.

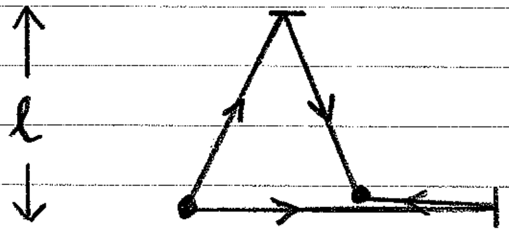
Result of experiment

Nothing. No change in interference pattern. Tried for several days, as ⊕ rotated on axis. Nothing.

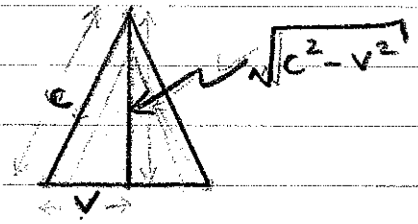
It was as if the aether moved with ⊕.

Classical expectation of outcome of Michelson - Morley experiment

Path of light through aether :



Path ↗↘



Distance ↗ $= l \dfrac{c}{\sqrt{c^2-v^2}}$

Distance ↘ = same

⇒ total distance ↗↘ $= 2l \dfrac{c}{\sqrt{c^2-v^2}}$

$$\Rightarrow \boxed{\text{Total time } \nearrow\searrow \; \cancel{d=} \; \frac{\text{distance}}{\cancel{\text{velo}}\;\text{velocity} \; \cancel{\text{car}} = c} = \cancel{\frac{2l}{c\sqrt{c^2-v^2}}} = \frac{2l}{\sqrt{c^2-v^2}}}$$

Path →←

Let time → $\equiv t_{\rightarrow}$.

Then distance → $= ct_{\rightarrow}$ how far light goes

$= l + vt_{\rightarrow}$ distance l plus distance moved by observer

Whence $t_{\rightarrow} = \dfrac{l}{c-v}$

Similarly let time $\leftarrow \equiv t_{\leftarrow}$.

Then distance $\leftarrow$ $= ct_{\leftarrow}$ how far light goes

$= \ell - vt_{\leftarrow}$ distance ℓ minus distance moved by observer

whence $t_{\leftarrow} = \dfrac{\ell}{c+v}$

$$\begin{aligned} \boxed{\text{Total time } \rightleftarrows} &= t_{\rightarrow} + t_{\leftarrow} \\ &= \frac{\ell}{c-v} + \frac{\ell}{c+v} \\ &\boxed{= \frac{2\ell c}{c^2-v^2}} \end{aligned}$$

Conclusion

$$\boxed{\frac{\text{Time } \wedge}{\text{Time } \rightleftarrows}} = \frac{\dfrac{2\ell}{\sqrt{c^2-v^2}}}{\dfrac{2\ell c}{c^2-v^2}}$$

$$\boxed{= \sqrt{1 - \frac{v^2}{c^2}}}$$

takes slightly less time to go $\wedge$ than $\rightleftarrows$.

George Fitzgerald, Irish (1889)

Suggested MM experiment would be understood if all lengths along direction of motion through aether were contracted by $\sqrt{1-\frac{v^2}{c^2}}$.

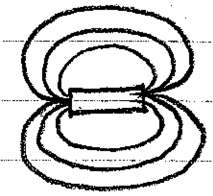
= "Fitzgerald contraction".

Hendrick Lorentz,	Joseph Larmor,	Henri Poincaré
Dutch	English	French

Maxwell's equations looked simple in frame stationary with respect to aether, but became complicated in moving frame.

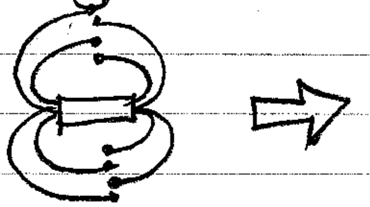
Example

Static frame



magnetic field has no ends

Moving frame



magnetic field acquires ends

//

Lorentz, Larmor, Poincaré independently noticed that if Maxwell's equations were taken to be same in moving frame as in static frame, then

- Maxwell's equations looked much more elegant
- it would account for failed Michelson-Morley experiment:
  - lengths would be contracted along direction of motion through aether;
  - clocks moving through aether would run slow.

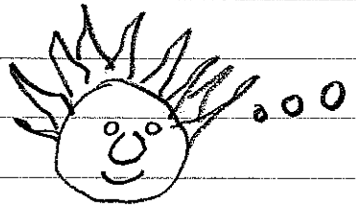
Result: "Lorentz transformations" of spacetime.

Revised Maxwell's equations,
combined with new "Lorentz law" for motion of charges
were Lorentz covariant
NOT Galilean covariant.

equations took same form under Lorentz transformations.

Tried to explain how interaction of objects with aether might lead to these effects.
Didn't click that solution was to abolish the aether — abolish absolute spacetime.

Einstein's (1905) Theory of Special Relativity



There is no absolute spacetime.
Spacetime is relative to the observer.

SR based on 4 postulates:

0. Spacetime forms a 4-dimensional continuum.
1. The existence of globally inertial frames.
2. The speed of light is a universal constant, the same for any observer.
3. The Principle of Special Relativity.

the technical mathematical term

0. Spacetime forms a 4-dimensional manifold

- Postulate inherited from classical concept of 3D space
  1D time

- Physicists widely expect this assumption to break down on very small scales. Expect spacetime to become quantized on scales of about

$$\text{Planck length} \equiv \left(\frac{G\hbar}{c^3}\right)^{\frac{1}{2}} \approx 10^{-35} \text{ m}$$

$$\text{Planck time} \equiv \left(\frac{G\hbar}{c^5}\right)^{\frac{1}{2}} \approx 10^{-43} \text{ s}$$

1. Existence of globally inertial frames

- Postulate carried over from classical mechanics (Newton's 1st law of motion)

- Statement:
  there exist global spacetime frames (ie. coordinate systems) with respect to which unaccelerated objects move in straight lines at constant velocity.
- Implicit consequence:
  spacetime geometry is flat, Euclidean.
  Postulate breaks down in general relativity.

- Mathematically:
  transformation of spacetime interval from one inertial frame to another is $\boxed{\text{linear}}$

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = L \begin{pmatrix} t \\ x \end{pmatrix}$$

spacetime interval in another frame

spacetime interval in one frame

Lorentz transformation is linear (a matrix).

2. The speed of light c is a universal constant

Q: What does it mean to say that a wave travels at a certain velocity?

Q: What is waving?

- Destroys classical Galilean spacetime, because velocity (of light, in particular) can no longer be additive (as we'll see).
- Mathematically: crucial postulates which leads to specific form of Lorentz transformations (as well see).

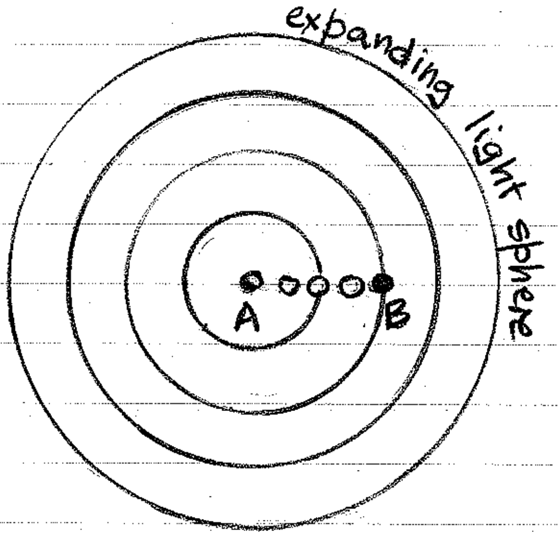
3. The Principle of Relativity

- Statement: The laws of physics are the same in any inertial frame, regardless of position or velocity.
- Physically: Absolute spacetime does not exist. Only relative spacetime intervals between observer are measureable.

- Mathematically:
  The laws of physics must be "Lorentz covariant".

- Note that
  Principle of Relativity $\not\Rightarrow$ constancy of c although they are consistent; and
  constancy of c $\not\Rightarrow$ Principle of Relativity although former was the inspiration of latter.

Paradox of the constancy of the speed of light



Light emitted at point A.

A's point of view:
c is constant, so light sphere expands uniformly,
A is at center of light sphere.

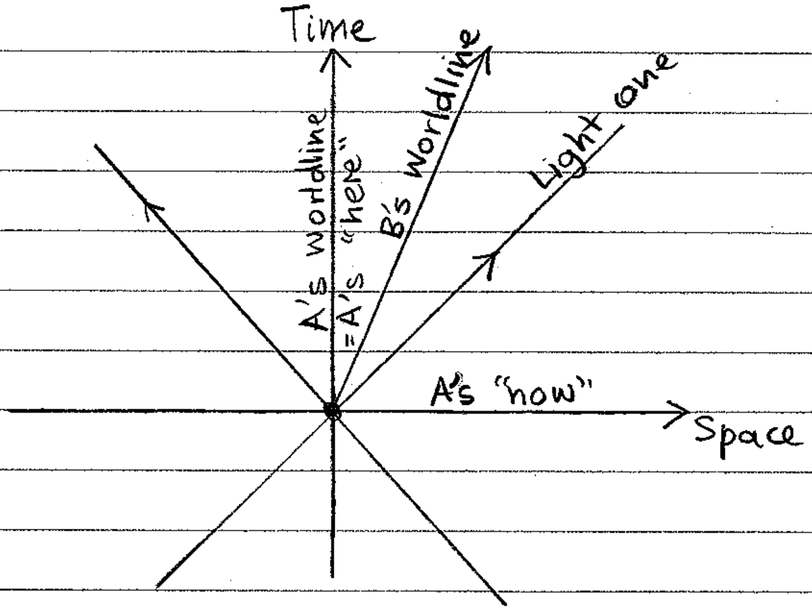
B starts at point A at instant of emission,
but is moving relative to A.

B's point of view:
c is constant, etc, so
B is at center of light sphere.

Paradox!

Objection: It was A that emitted the light,
so of course B doesn't have to be at center.

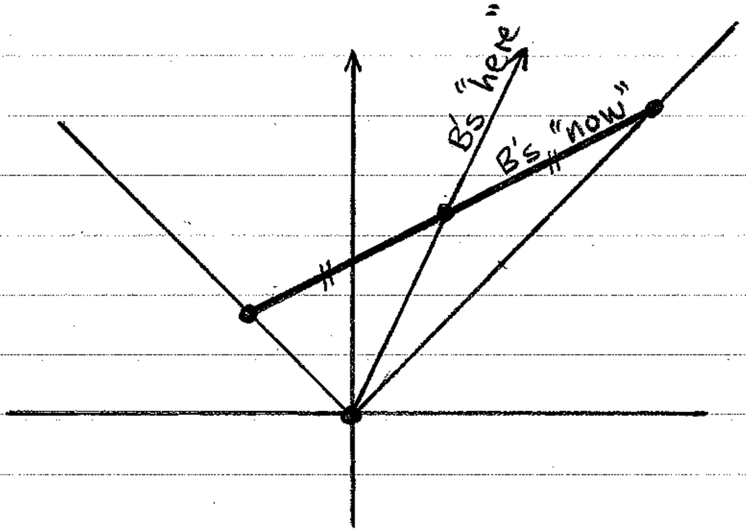
Spacetime diagram



Spacetime diagram uses units of space & time so that $c = 1$; hence light always moves at 45° from vertical.

Each point in spacetime is called an $\boxed{\text{event}}$.

Solution to paradox of constant c



B's idea of "here" and "now" is different from A's.

$\mathbf{x}$ = constant

t = constant

"hypersurface of simultaneity"

B is at center of light cone from own point of view.

Transformation between A & B frames is called "Lorentz transformation".

Challenge:

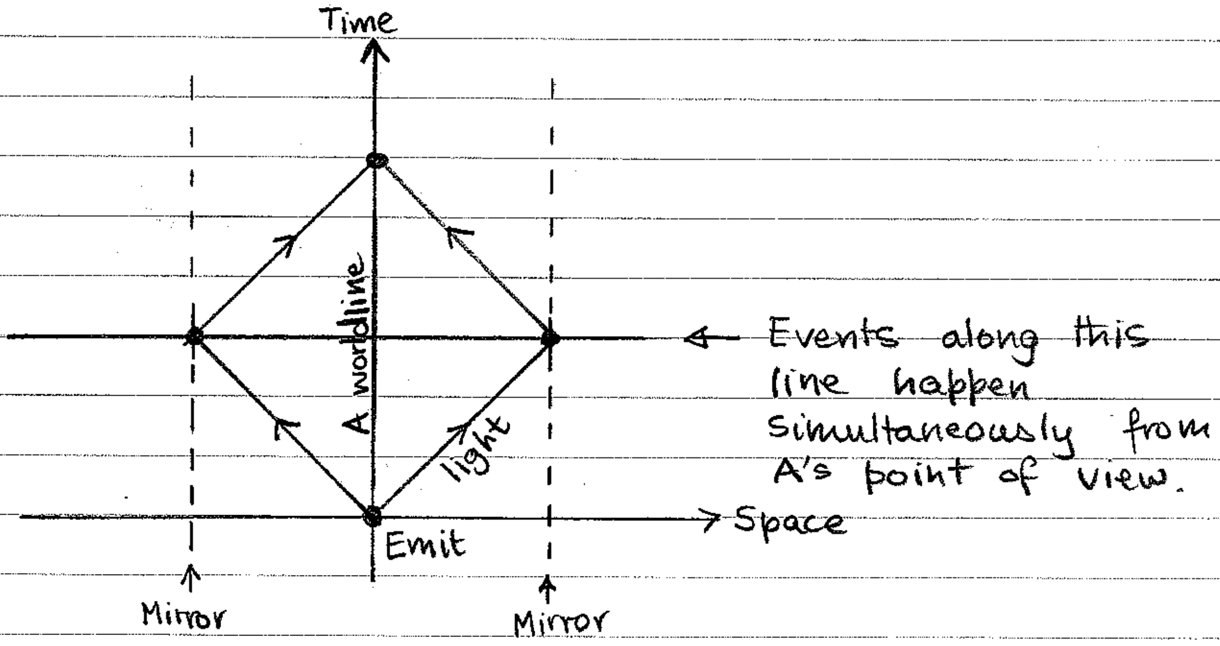
Now go home and figure out the mathematics.

Simultaneity

How to define simultaneous?

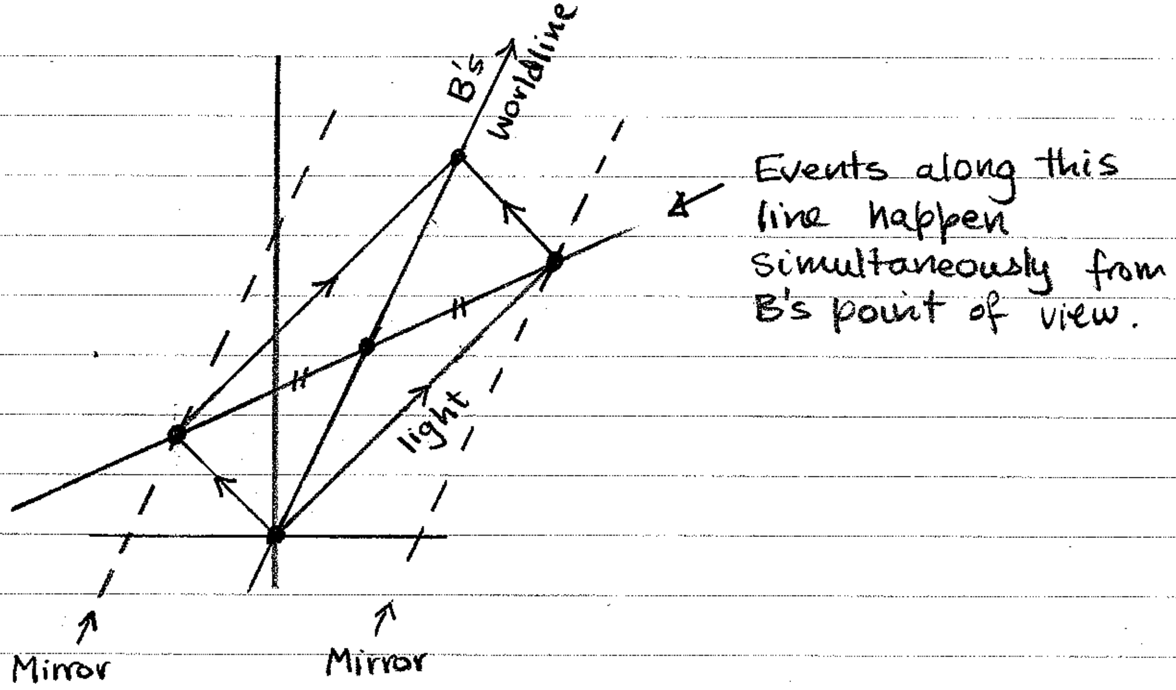
Experiment:

A (or B) sets mirrors equidistant from self. Knows equidistant if emitted light pulse returns at same instant.



Events along this line happen simultaneously from A's point of view.

This is a spacetime diagram of A's experiment.



This is a spacetime diagram of B's experiment

Challenge:
Now go home and figure out the mathematics.

Derivation of Lorentz transformation (geometric)

1. Draw axes of spacetime diagram. Take $c = 1$.
2. Draw worldline of B moving at v in x-direction.
3. Construct light rectangle with sides at 45° from vertical, B at center.
4. Spacetime coordinates of B are (assert)

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma \\ \gamma v \\ 0 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

in A frame   in B frame

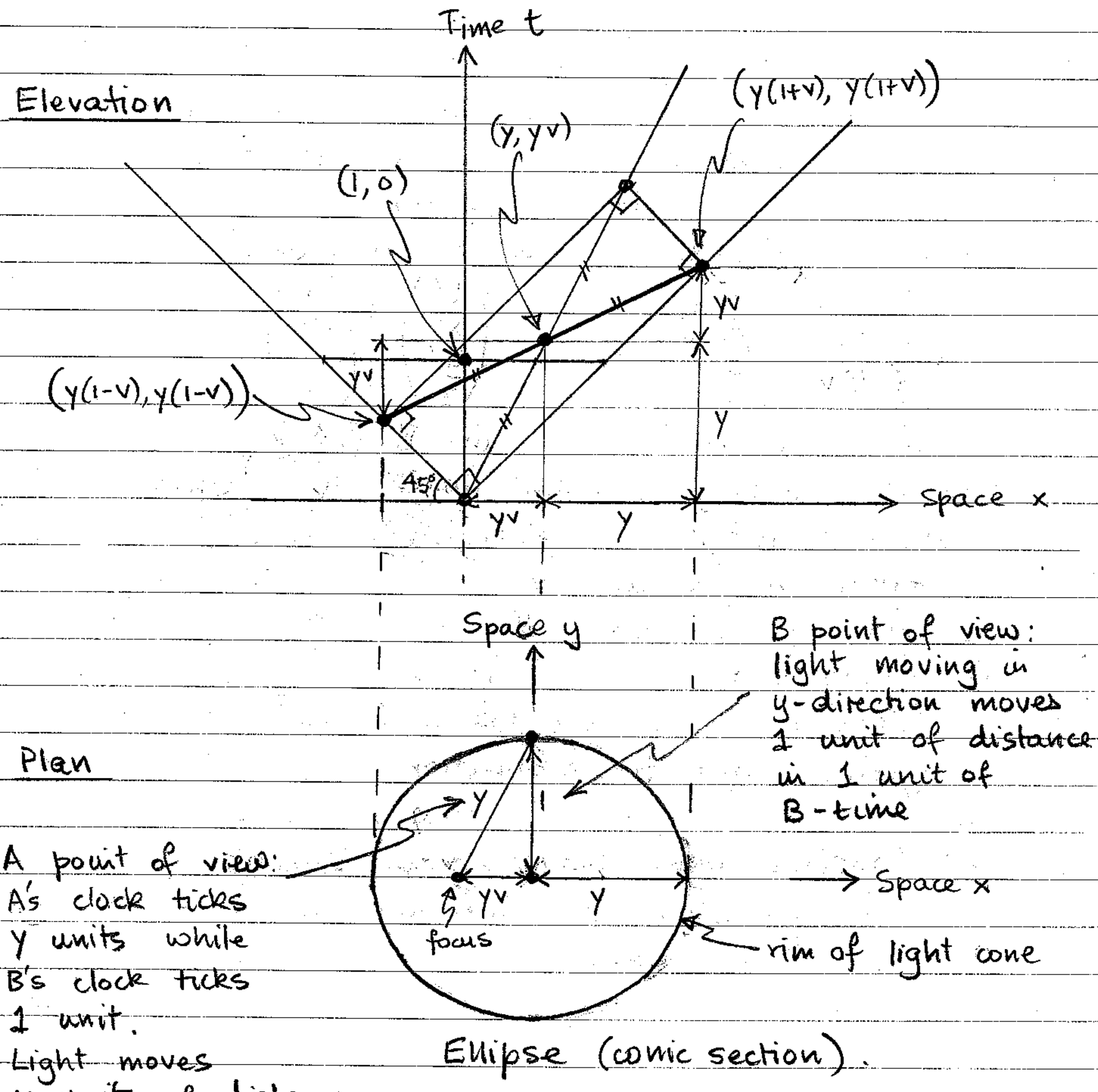
relative to origin, event where light emitted.

5. Figure out lengths of everything in sight in terms of γ and v.
6. Determine γ in terms of v
   (a) Lorentz transformation at v followed by Lorentz transformation at $-v$ yields unit transformation

   or,

   (b) Consider light traveling in direction perpendicular to direction of v
   e.g. y-direction.

Units so $c = 1$. Coordinates (t, x).



B point of view: light moving in y-direction moves 1 unit of distance in 1 unit of B-time

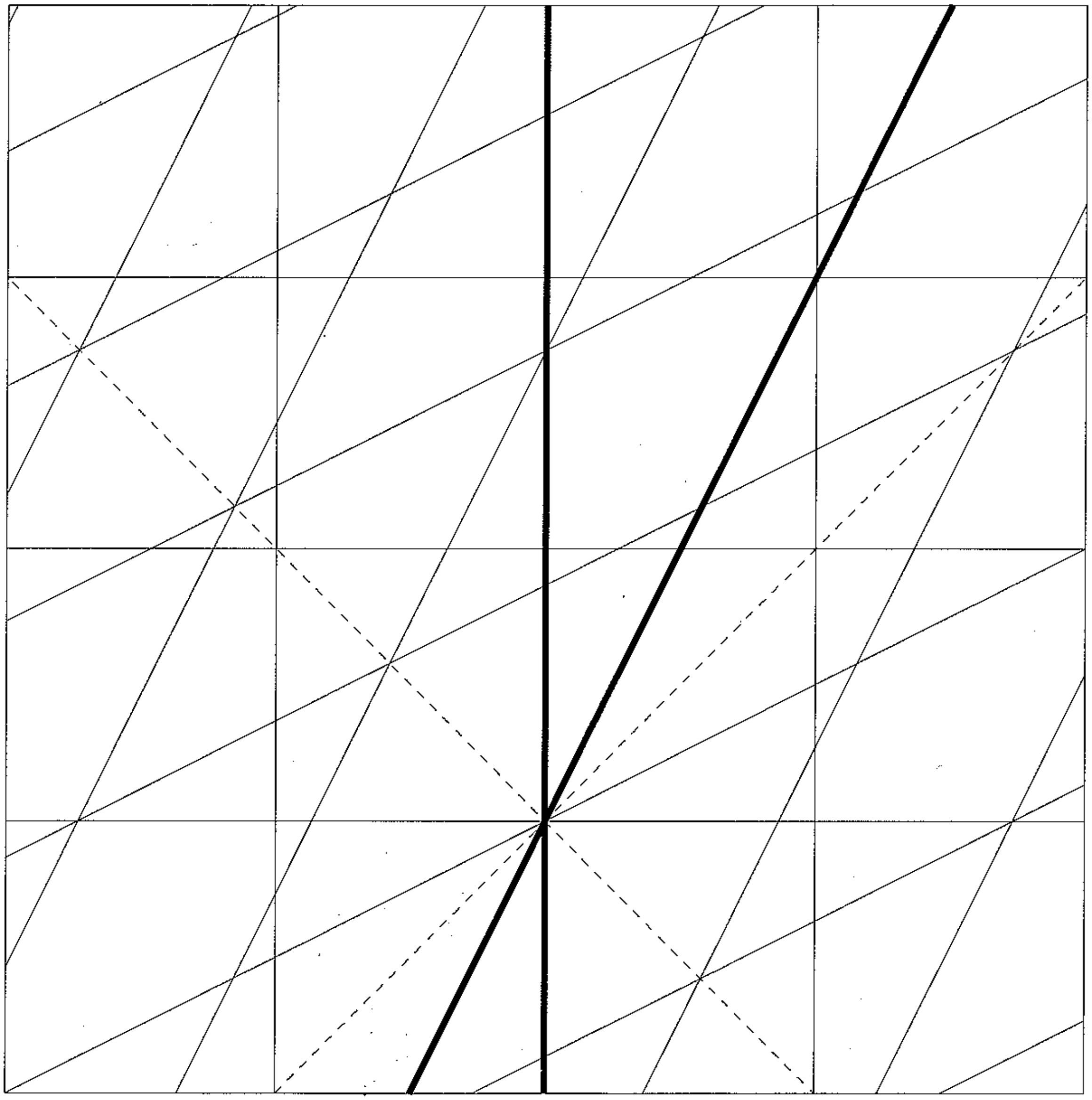
A point of view: A's clock ticks y units while B's clock ticks 1 unit. Light moves y units of distance in y unit of time.

Ellipse (conic section).

Math: $1 + y^2v^2 = y^2$ (Pythagoras)

whence $\boxed{y = \dfrac{1}{\sqrt{1-v^2}}}$

1.38



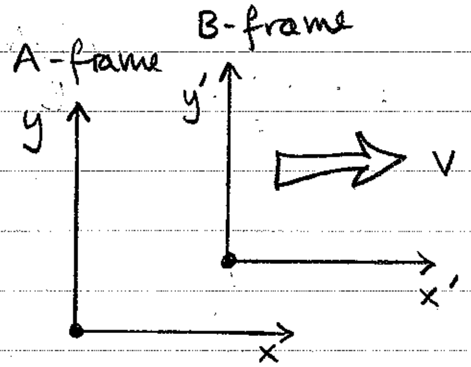
Derivation of Lorentz transformation (more formal)

(1) Postulate of existence of globally inertial frames

$\Rightarrow$ straight lines in one inertial frame transform to straight lines in another

$\Rightarrow$ Lorentz transformations are linear.

Take B frame (primed) to be moving at velocity v along x-direction relative to A frame (unprimed)



Lorentz transformation takes form

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \kappa & \lambda & 0 & 0 \\ \mu & \nu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix}$$

(2) Postulate of constancy of speed of light

$\Rightarrow$ (i) line $x = t$ in one frame $(c = 1)$ transforms to line $x' = t'$ in another

and similarly

(ii) line $x = -t$ in one frame transforms to line $x' = -t'$ in another

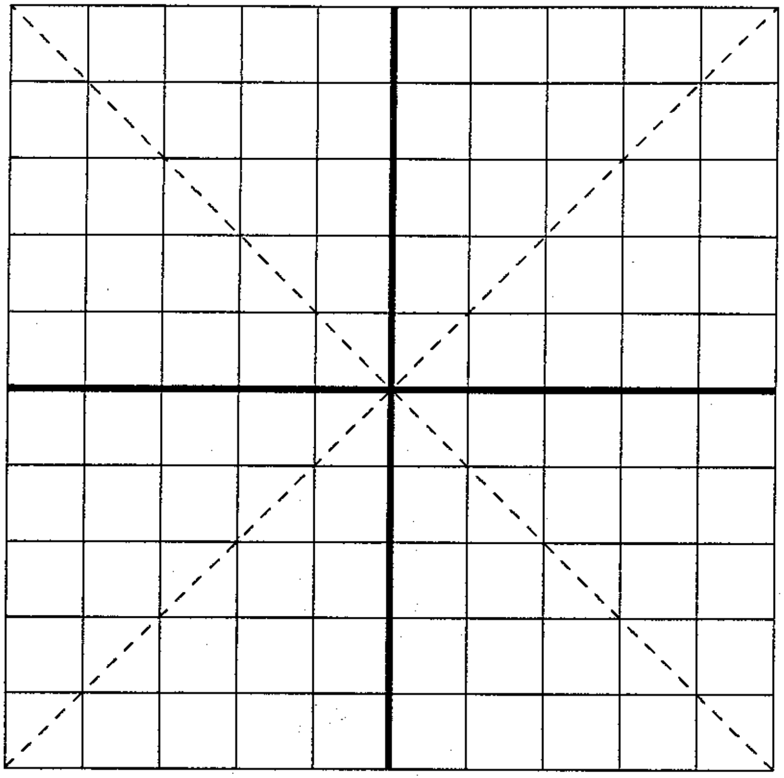
Math:

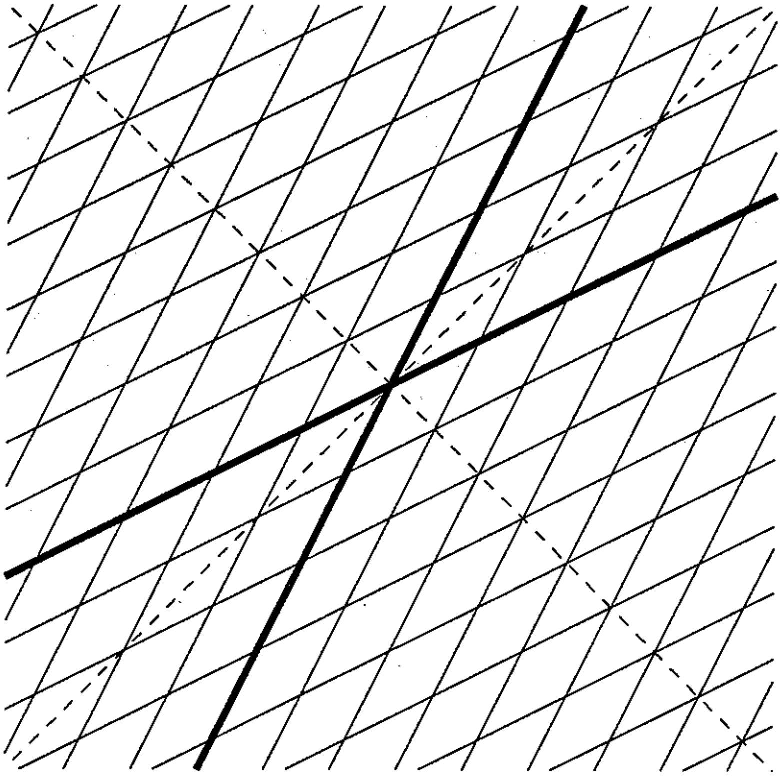
(i) $\Rightarrow \quad x = (\mu + \nu)t' = t = (\kappa + \lambda)t'$

(ii) $\Rightarrow \quad x = (\mu - \nu)t' = -t = -(\kappa - \lambda)t'$

together $\Rightarrow \quad \kappa = \nu$ and $\lambda = \mu$

so $\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \kappa & \lambda \\ \lambda & \kappa \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$





(3) Consider event $\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in B frame.

This is spacetime interval moved by B in 1 tick of B's clock in B's frame.

By assumption B is moving at v in A's frame, which means $x = vt$ in A frame, by definition of velocity.

Math:

$$\begin{pmatrix} t \\ x \end{pmatrix} = \begin{pmatrix} \kappa & \lambda \\ \lambda & \kappa \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \kappa \\ \lambda \end{pmatrix}$$

$$= \gamma \begin{pmatrix} 1 \\ v \end{pmatrix}$$

some constant, a function of v

so $\begin{pmatrix} t \\ x \end{pmatrix} = \gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$

(4) What is γ?

(a) By isotropy of space, γ must be function of magnitude (not direction) of v;

(b) Lorentz transformation at v followed by Lorentz transformation at $-v$ must yield unit transformation.

ie, opposite direction

Math:

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} = \gamma \begin{pmatrix} 1 & -v \\ -v & 1 \end{pmatrix} \gamma \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

$$= \gamma^2 \begin{pmatrix} 1-v^2 & 0 \\ 0 & 1-v^2 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \end{pmatrix}$$

$$\Longrightarrow \quad \boxed{\gamma = \frac{1}{\sqrt{1-v^2}}} \qquad (c = 1)$$

Notice that determinant of Lorentz transformation is 1

$$\begin{vmatrix} \gamma & \gamma v \\ \gamma v & \gamma \end{vmatrix} = 1$$

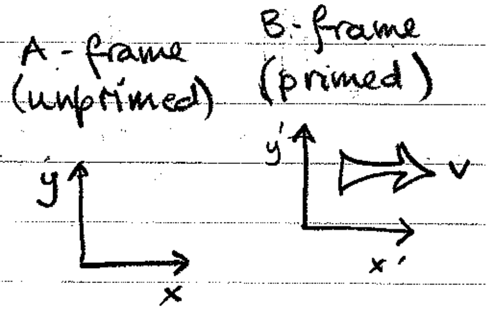
which makes sense.

4-vectors

A 4-vector is a spacetime interval $\begin{pmatrix} t \\ \mathbf{r} \end{pmatrix}$ which transforms according to Lorentz transformation under "Lorentz boost", ie change in velocity of frame of reference. Thus

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma & \gamma v & 0 & 0 \\ \gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix}$$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$



Invariant spacetime interval (scalar)

Quantity

$$s^2 = -t^2 + r^2$$
$$= -t^2 + (x^2 + y^2 + z^2)$$

Proof: Problem Set

remains unchanged under Lorentz transformation. Such quantities are called scalars.

Single scalar spacetime interval s replaces two scalar quantities

t time interval

$r = \sqrt{x^2 + y^2 + z^2}$ distance interval

of classical Galilean spacetime.

Note

Some authors prefer convention

$$s^2 = t^2 - r^2$$

which is negative of above.

Jargon

Spacetime vector $(t, \mathbf{r})$ is called

timelike if $s^2 < 0$

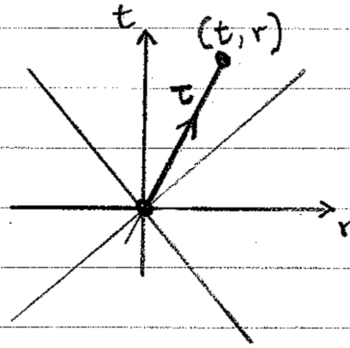
null (or lightlike) " $s^2 = 0$

spacelike $s^2 > 0$

Proper time, distance

Proper time along time-like $(t > r)$ interval

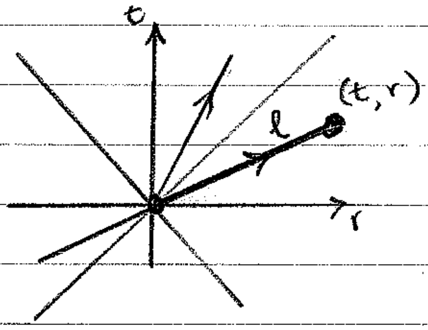
$$\tau \equiv \sqrt{t^2 - r^2}$$



is the time experienced by observer moving along that interval

Proper distance along space-like $(t < r)$ interval

$$l \equiv \sqrt{r^2 - t^2}$$



is the distance between two events measured by an observer for whom those events are simultaneous.

Relation between proper time τ and time t

If interval corresponds to motion at velocity v, ie. if $r = vt$, then

$$\tau = \sqrt{t^2 - r^2} = t\sqrt{1 - v^2}$$

$$= \frac{t}{\gamma}$$

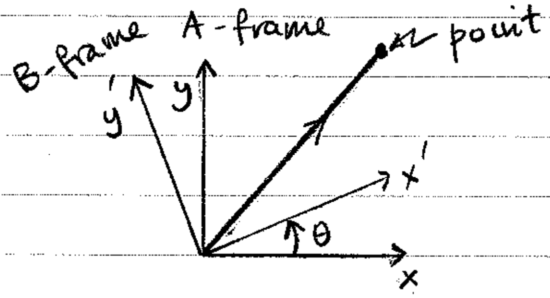
This is famous Lorentz time dilation: the proper time experienced by moving observer is factor γ less than according to onlooker.

Lorentz boost as rotation by imaginary angle

Time can often be treated mathematically as an imaginary ($i = \sqrt{-1}$) spatial dimension. Thus, e.g.,

$$s^2 = -\left[(it)^2 + x^2 + y^2 + z^2\right]$$

Consider a spatial rotation, where B frame is rotated by angle θ about z-axis relative to A frame.



Then

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

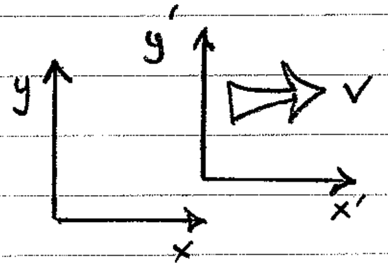
$$\underset{\substack{\uparrow \\ \text{coords of point} \\ \text{in B-frame}}}{\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \underset{\substack{\uparrow \\ \text{coords of point} \\ \text{in A-frame}}}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}}$$

Recall

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \qquad \sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2}$$

$$= \cos i\theta \qquad\qquad = -i\sin i\theta$$



Lorentz boost in x-direction can be regarded as rotation by imaginary angle $i\theta$ in $t-x$ plane.

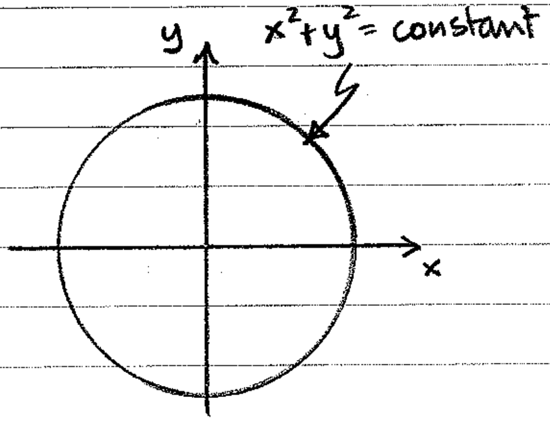
Thus

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cosh\theta & \sinh\theta & 0 & 0 \\ \sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix}$$

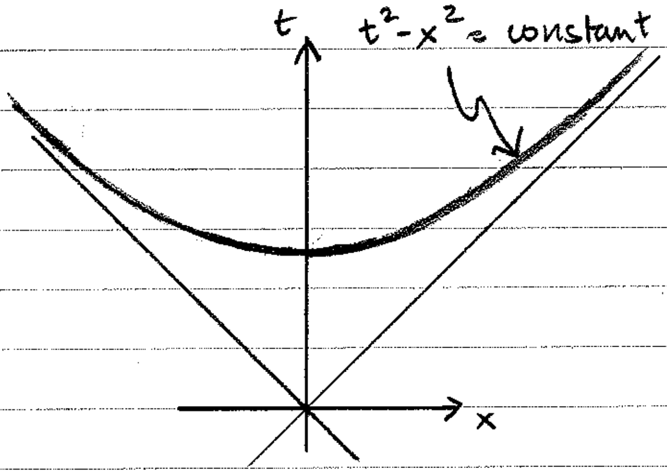
$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\theta & -\sinh\theta & 0 & 0 \\ -\sinh\theta & \cosh\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Have

$$\gamma = \cosh\theta$$
$$\gamma v = \sinh\theta$$
$$v = \tanh\theta$$



Trajectory of point under spatial rotation by various angles.



Trajectory of point under Lorentz boost by various velocities.

The usual relativistic phenomena ...

Time dilation

Friend moves at v relative to me.
I think friend's clock ticks slower by
Lorentz factor $\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$

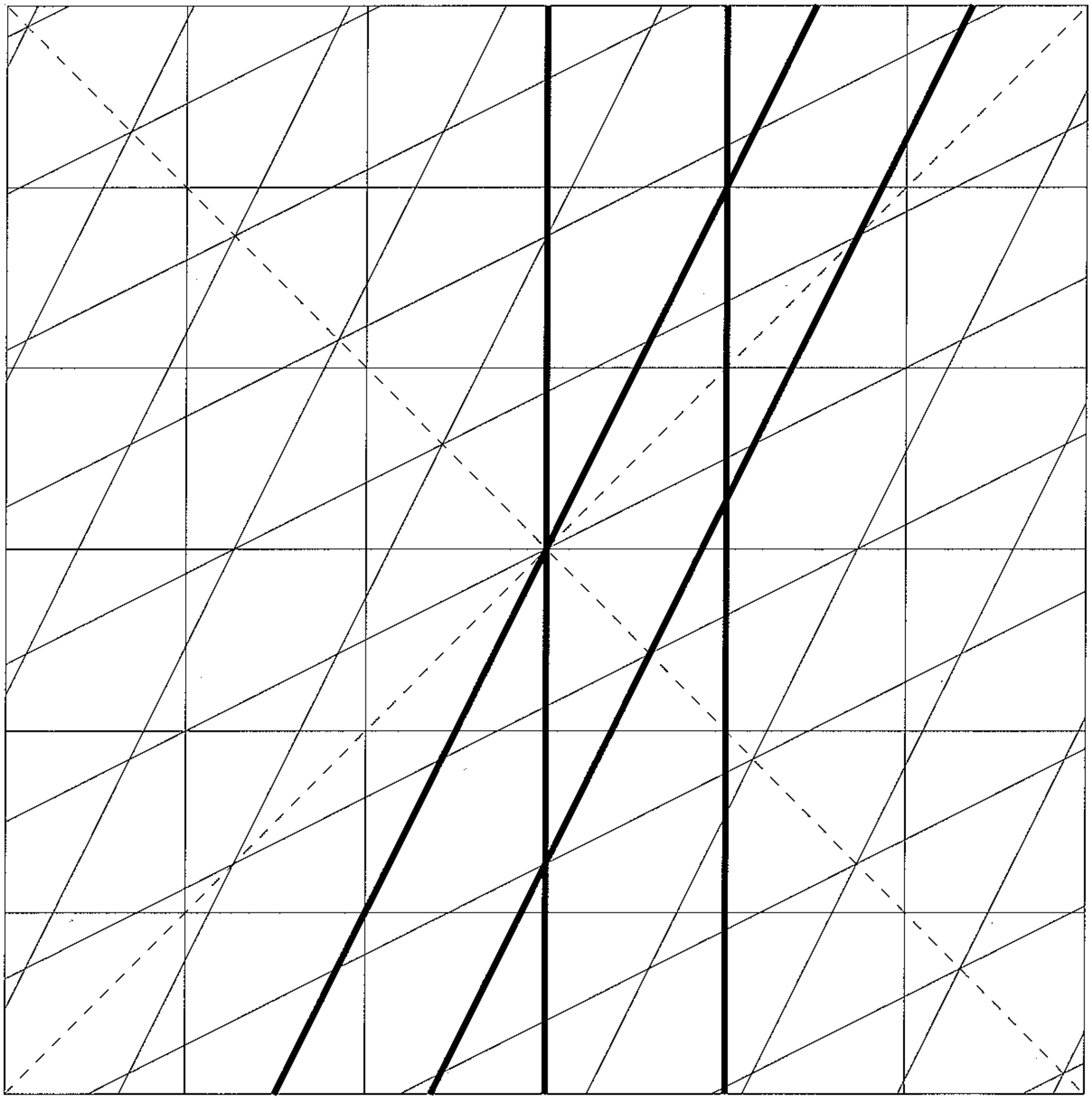
Paradox: I think friend's clock ticks slow;
friend thinks my clock ticks slow;
how can that be?

Fitzgerald - Lorentz contraction

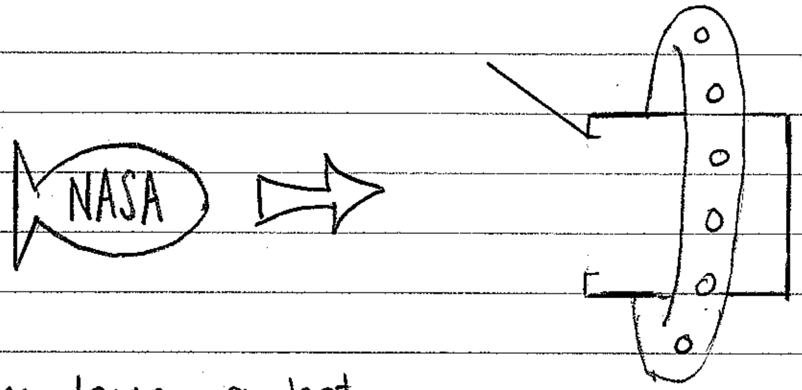
Friend moves at v relative to me.
I think friend is contracted by
Lorentz factor γ along direction of motion.

Paradox: I think friend is contracted;
friend thinks I am contracted;
how can that be?

1.48



"Pole-in-the-barn" paradox

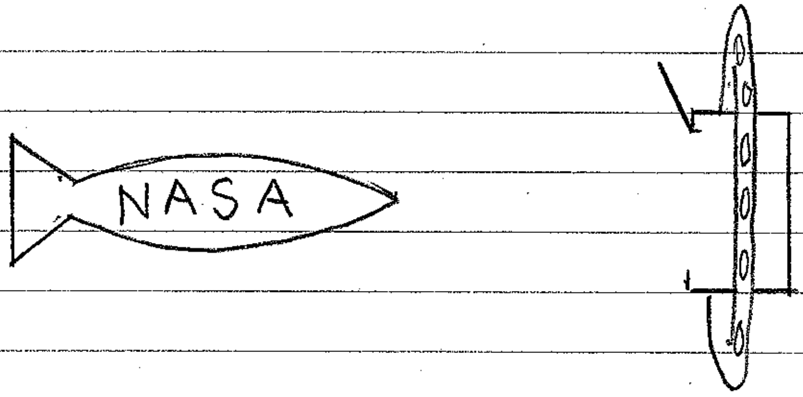


200 m long rocket
appears Lorentz-contracted to 100 m

100 m long spacestation

Will rocket make it inside spacestation?

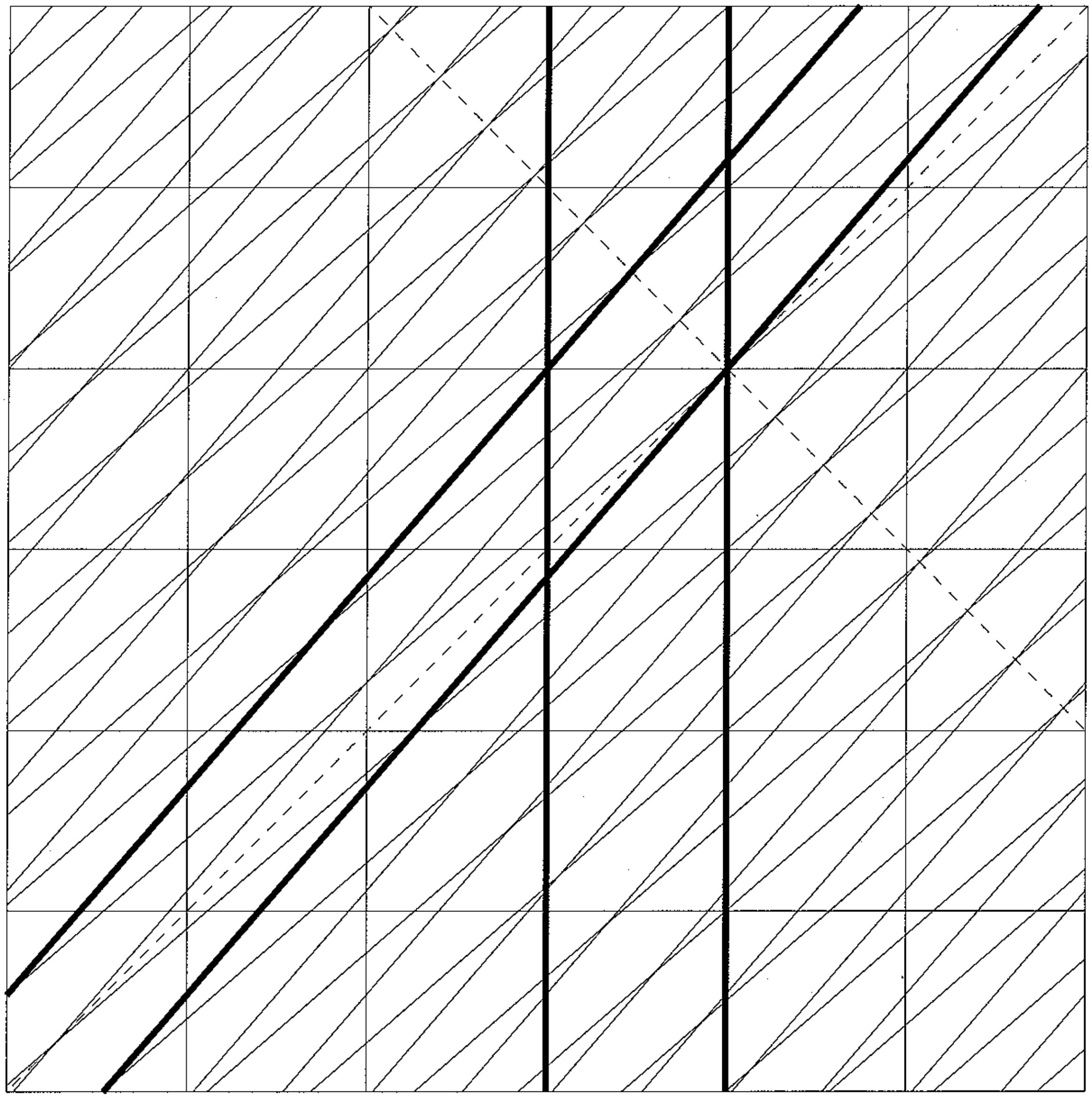
Of course from rocket's point of view:



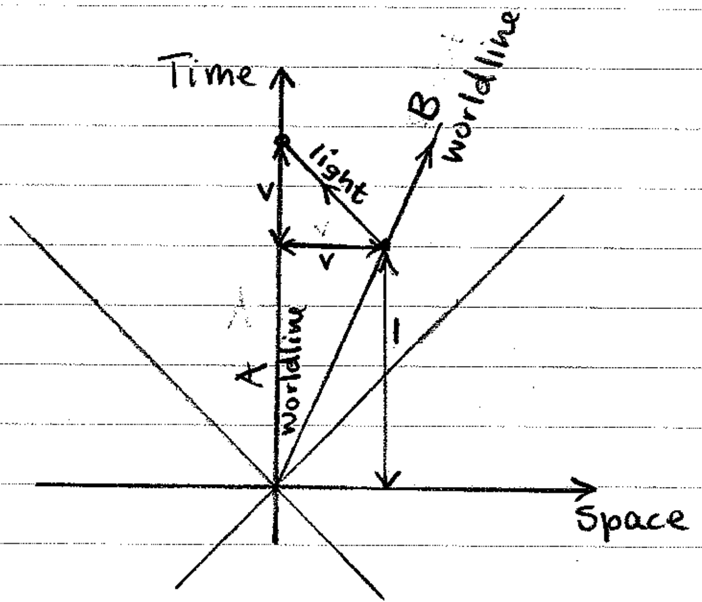
200 m long rocket

100 m long spacestation
appears Lorentz-contracted to 50 m.

1.50



Apparently velocity on trip to α Cen



A's view:

B moves distance $\Delta x = v$ in time $\Delta t = 1$.

But takes extra time Δt_{travel} for light signal to return.

So actually looks like
B moves distance $\Delta x = v$
in time $\Delta t_{obs} = \Delta t + \Delta t_{travel}$
$= 1 + v$

ie
Apparent velocity is

$$\boxed{v_{obs} = \frac{v}{1+v}}$$

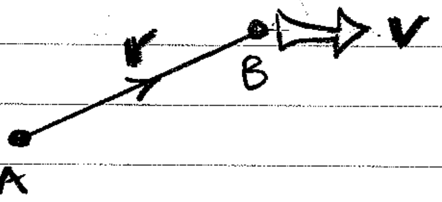
Examples

1. If $\underset{\text{outward}}{v \approx 1}$, then $v_{obs} \approx \frac{1}{2}$
2. If $\underset{\text{inward}}{v \approx -1}$, then $v_{obs} \approx \frac{-1}{0} \to -\infty$

   Superluminal motion.

Problem (generalization to where v is not along line of sight)

Suppose that B is at position $\mathbf{r}$
relative to A, in A frame,
and B is moving at $\mathbf{v}$ relative to A.

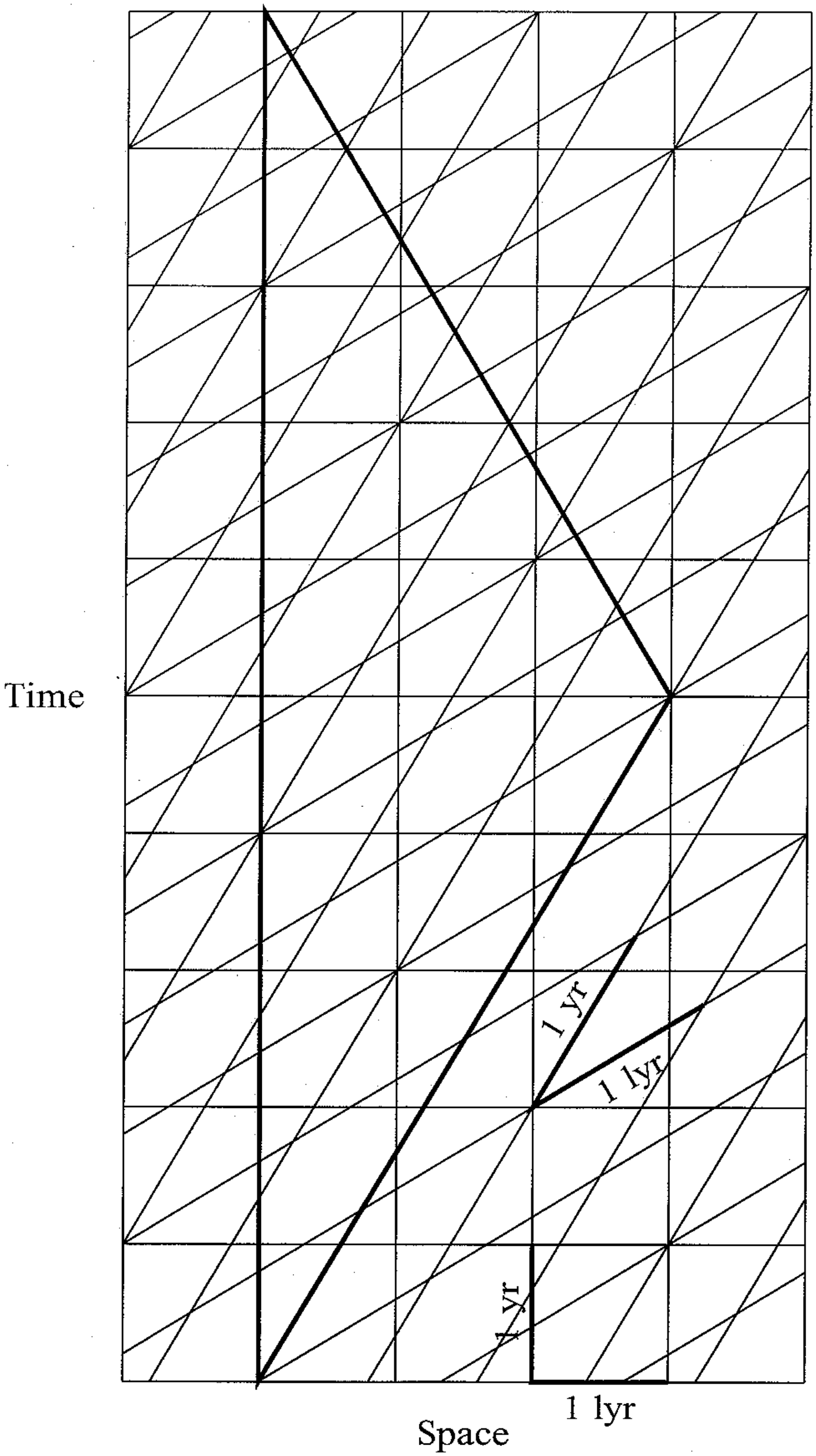


Show that

$$\mathbf{v}_{obs} = \frac{\mathbf{v}}{1+\mathbf{v}\cdot\underset{\uparrow}{\hat{\mathbf{r}}}}$$

unit vector in direction to B.

Twin Paradox



MATRICES REVIEW

Matrix

is a rectangular array of numbers

$$A \equiv \begin{array}{cc} & \xleftrightarrow{\qquad M \qquad} \\ \updownarrow L & \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1M} \\ A_{21} & A_{22} & \cdots & A_{2M} \\ \vdots & & & \\ A_{L1} & \cdots & \cdots & A_{LM} \end{pmatrix} \end{array} \quad \text{is } \overset{\text{columns}}{M} \times \overset{\text{rows}}{L} \text{ matrix}$$

Index notation;

$$A_{ij} \qquad \begin{aligned} i &= 1, \dots, L \\ j &= 1, \dots, M \end{aligned}$$

A vector can be regarded as a matrix with just 1 row or column

Matrix Addition

defined in the obvious way

$$A + B \equiv \begin{array}{cc} & \xleftrightarrow{\; M \;} \\ \updownarrow L & \begin{pmatrix} A_{11} & \cdots & A_{1M} \\ \vdots & & \vdots \\ A_{L1} & \cdots & A_{LM} \end{pmatrix} \end{array} + \begin{array}{cc} & \xleftrightarrow{\; M \;} \\ \updownarrow L & \begin{pmatrix} B_{11} & \cdots & B_{1M} \\ \vdots & & \vdots \\ B_{L1} & \cdots & B_{LM} \end{pmatrix} \end{array}$$

$$\equiv \begin{array}{cc} & \xleftrightarrow{\qquad\qquad M \qquad\qquad} \\ \updownarrow L & \begin{pmatrix} A_{11}+B_{11} & A_{12}+B_{12} & \cdots & A_{1M}+B_{1M} \\ \vdots & \vdots & & \\ A_{L1}+B_{L1} & - & - & A_{LM}+B_{LM} \end{pmatrix} \end{array}$$

In index notation

$$(A+B)_{ij} \equiv A_{ij} + B_{ij}$$

Matrix addition has usual properties:

- associative $(A+B)+C = A+(B+C)$.
- commutative $A+B = B+A$.

Matrix Multiplication

$$C = A\ B$$

$$L\updownarrow\overset{\longleftarrow N \longrightarrow}{\begin{pmatrix} C_{11} & \cdots & C_{1N} \\ \vdots & & \\ C_{L1} & \cdots & C_{LN} \end{pmatrix}} = L\updownarrow\overset{\longleftarrow M \longrightarrow}{\begin{pmatrix} A_{11} & \cdots & A_{1M} \\ \vdots & & \vdots \\ A_{L1} & \cdots & A_{LM} \end{pmatrix}} \; M\updownarrow\overset{\longleftarrow N \longrightarrow}{\begin{pmatrix} B_{11} & \cdots & B_{1N} \\ \vdots & & \\ B_{M1} & \cdots & B_{MN} \end{pmatrix}}$$

by definition means

$$\boxed{C_{ik} = \sum_{j=1}^{M} A_{ij} B_{jk}} \qquad \begin{array}{l} i = 1 \ldots L \\ k = 1 \ldots N \end{array}$$

ie C_{ik} is the scalar product of
the i'th row of A
with the k'th column of B.

Matrix multiplication has properties:

- associative $(AB)C = A(BC)$
- NOT commutative $AB \neq BA$ in general.

Implicit summation convention

Summation sign $\sum$ often omitted.
Summation over repeated indices implicitly assumed
ie

$$C_{ik} = A_{ij} B_{jk}$$

Example

$$\underset{\boxed{\,}}{y_i} \;\underset{=}{=}\; \underset{\boxed{\quad\;}}{A_{ij}}\, \underset{\boxed{\,}}{x_j} \qquad \left(\equiv \sum_{j=1}^{M} A_{ij}\, x_j \right)$$

is equivalent to the set of linear simultaneous equations

$$\begin{array}{lcl} y_1 & = & A_{11} x_1 + A_{12} x_2 + \ldots + A_{1M} x_M \\ y_2 & = & A_{21} x_1 + A_{22} x_2 + \ldots + A_{2M} x_M \\ \vdots & & \qquad\qquad\qquad\qquad\qquad\quad \vdots \\ y_L & = & A_{L1} x_1 + A_{L2} x_2 + \ldots + A_{LM} x_M \end{array}$$

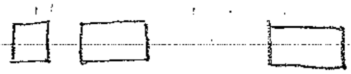
Unit matrix

$$\underline{1} \equiv \begin{array}{cc} & \xleftrightarrow{\quad L \quad} \\ \updownarrow L & \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & & \ddots & \\ 0 & \cdots & & & 1 \end{pmatrix} \end{array}$$

square matrix with $\underline{1}$'s on diagonal, 0's elsewhere.

Unit matrix has property:

$$\underline{1} A = A \quad , \qquad A \underline{1} = A$$



for all matrices A.

Inverse matrix

A^{-1} of square matrix

satisfies $A^{-1} A = A A^{-1} = \underline{1}$

IF inverse exists (may not).

If it does, then solution of

$$A x = y$$

is

$$x = A^{-1} y$$

Tensor

Is an array of numbers with several indices

A is scalar = tensor of rank 0
A_i " vector = " " " 1
A_{ij} " matrix = " " " 2
A_{ijk} = tensor of rank 3
$\vdots$
etc.

Implicit summation convention for repeated indices is again widely used.

4-vector index notation

In special & general relativity it is convenient to introduce 2 versions of same 4-vector quantity

raised index

$$r^i \equiv (t, \mathbf{r}) \equiv (t, x, y, z)$$

are "contravariant" components of the 4-vector while

lowered index

$$r_i \equiv (-t, \mathbf{r}) \equiv (-t, x, y, z)$$

negative sign introduced here

are "covariant" components of the 4-vector,

forget this dumb notation!

Indices i run over 0, 1, 2, 3
so $(r^0, r^1, r^2, r^3) = (t, x, y, z)$

Why?
Because

$$r^i r_i \left(\equiv \sum_{i=0}^{3} r^i r_i \right) = r_i r^i$$

$$= r^0 r_0 + r^1 r_1 + r^2 r_2 + r^3 r_3$$

$$= -t^2 + x^2 + y^2 + z^2$$

is Lorentz scalar.

In SR & GR, one of pair of repeated indices in always raised, other lowered.

Energy - Momentum 4-vector

Symmetry argument

Translation symmetry	⇒	Conservation law
Time		Energy
Space		Momentum

suggests

$$\left.\begin{array}{l} \text{Energy} = \text{time component} \\ \text{Momentum} = \text{space component} \end{array}\right\} \text{of 4-vector.}$$

Principle of Special Relativity

Energy
Momentum = constant

should take same form in any inertial frame.

Equation should be Lorentz - covariant
ie. transform like a Lorentz 4-vector

or tensor, more generally.

Constuction of energy-momentum 4-vector

Require:

1. it's a 4-vector
2. goes over to Newtonian limit as $v \to 0$

Newtonian limit:

$$\underset{\substack{\uparrow\\ \text{momentum}}}{\mathbf{p}} = \underset{\substack{\uparrow\\ \text{mass}}}{m}\,\underset{\substack{\nwarrow\\ \text{velocity}}}{\mathbf{v}} = m\frac{d\mathbf{r}}{dt}$$

4D version:

Need to do 2 things to Newtonian momentum:

- replace $\mathbf{r}$ by 4-vector $r^i = (t, \mathbf{r})$
- replace dt by a scalar –
  only obvious choice is proper time interval $d\tau$.

Result:

$$p^i = m\frac{dr^i}{d\tau}$$

$$= m\left(\frac{dt}{d\tau}, \frac{d\mathbf{r}}{d\tau}\right)$$

$$= m\left(\gamma, \gamma\mathbf{v}\right)$$

which are SR versions of Energy E & momentum $\mathbf{p}$

$$= (E, \mathbf{p})$$

Special relativistic energy

$$E = m\gamma \qquad \text{(units } c = 1\text{)}$$

or, restoring standard units,

$$E = mc^2\gamma$$

Try expanding γ for small velocity v:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = 1 + \frac{1}{2}\frac{v^2}{c^2} + \dots$$

so

$$E = mc^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} + \dots\right)$$

$$= \underset{\substack{\uparrow \\ \text{rest-mass} \\ \text{energy}}}{mc^2} + \underset{\substack{\uparrow \\ \text{nonrelativistic} \\ \text{kinetic energy.}}}{\frac{1}{2}mv^2} + \dots$$

Rest mass is a scalar

The scalar quantity constructed from the energy-momentum 4-vector $p^i = (E, \mathbf{p})$ is

$$p^i p_i = -E^2 + p^2$$

$$= m^2(-\gamma^2 + \gamma^2 v^2)$$

$$= -m^2 \qquad \text{minus from } -{+}{+}{+} \text{ signature}$$

is minus square of rest mass.

Photon energy-momentum

Photons have zero rest mass

$$m = 0 .$$

Thus $p^i p_i = -E^2 + p^2 = -m^2 = 0$

whence $p \equiv |\mathbf{p}| = E$

so

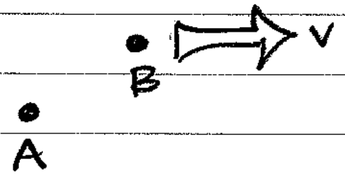
$$\boxed{p^i} = (E, \mathbf{p}) = E(1, \mathbf{n})$$

$$\boxed{= h\underset{\substack{\uparrow \\ \text{photon frequency}}}{\nu}(1, \mathbf{n})}$$

Photon velocity is $\mathbf{n}$, a unit vector.
Photon speed is one (the speed of light).

Lorentz transformation of photon energy-momentum

Follows usual rule.



Suppose B is moving relative to A at velocity v in x-direction.

Then

$$\underbrace{\begin{pmatrix} E' \\ p_x' \\ p_y' \\ p_z' \end{pmatrix}}_{\substack{\text{photon} \\ \text{4-momentum} \\ \text{in B frame}}} = \underbrace{\begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\substack{\text{Lorentz} \\ \text{transformation}}} \underbrace{\begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}}_{\substack{\text{photon} \\ \text{4-momentum} \\ \text{in A frame}}}$$

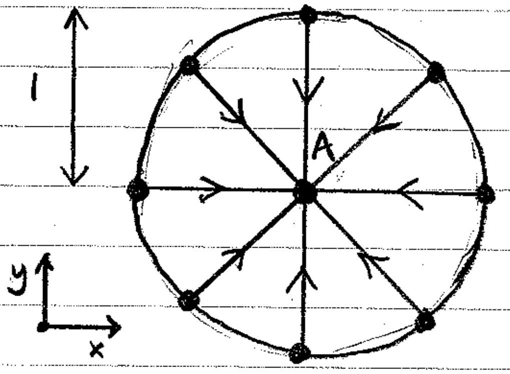
equivalently

$$\underset{\substack{\text{photon} \\ \text{frequency} \\ \text{in B-frame}}}{h\nu'} \begin{pmatrix} 1 \\ n_x' \\ n_y' \\ n_z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \overset{\substack{\text{photon frequency} \\ \text{in A frame}}}{h\nu} \begin{pmatrix} 1 \\ n_x \\ n_y \\ n_z \end{pmatrix}$$

$$= h\nu \begin{pmatrix} \gamma(1 - n_x v) \\ \gamma(n_x - v) \\ n_y \\ n_z \end{pmatrix}$$

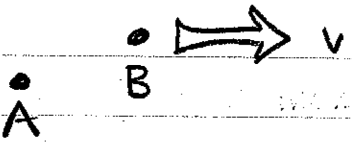
Relativistic beaming 1

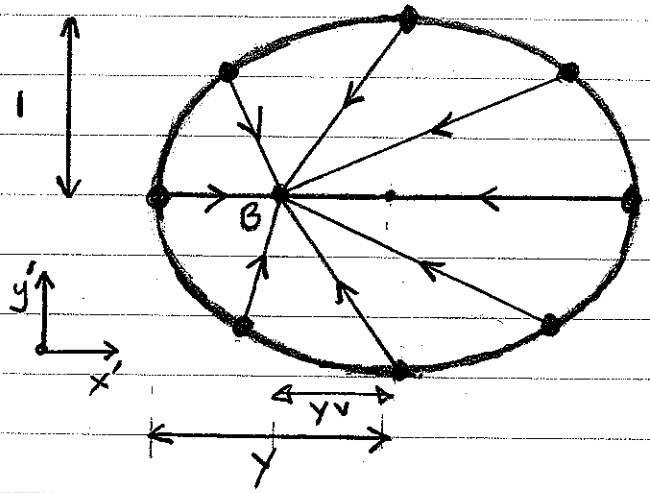
Suppose that A observes an isotropic landscape of photons, all same frequency



Arrows represent photon momenta in A frame.
Arrow length is photon energy.
(This is not a spacetime diagram)

B moves relative to A at velocity v in x-direction.
What does B see?





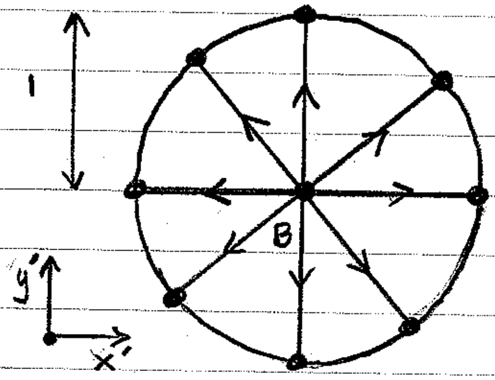
Arrows are photon momenta in B frame.
B is at focus of ellipse.

B sees photon landscape relativistically beamed

- concentrated in direction ahead
  like view seen through wide-angle lens
- blue-shifted ahead, redshifted behind.

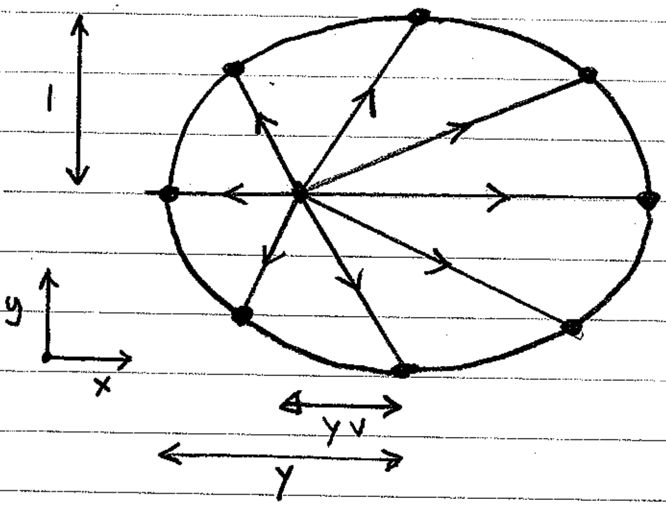
Relativistic beaming 2

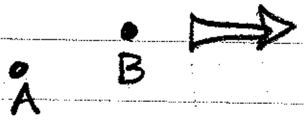
Suppose that B emits an isotropic landscape of photons, all same frequency



Arrows are photon momenta in B frame.

B moves relative to A at velocity v in x-direction. What does A think emission pattern is?





Arrows are photon momenta in A frame.

A thinks emission is relativistically beamed

- concentrated in the direction of motion of B
- blue-shifted ahead, redshifted behind.

Redshift

Astronomers define redshift z of photon by

$$z \equiv \frac{\lambda_{obs} - \lambda_{emit}}{\lambda_{emit}}$$

In relativity, it is often more convenient to use redshift factor $1+z$

$$\boxed{1+z \equiv \frac{\lambda_{obs}}{\lambda_{emit}}} = \frac{\nu_{emit}}{\nu_{obs}}$$

Special relativistic Doppler shift

Suppose (as usual) B (primed frame) is moving relative to A (unprimed frame) at v along x direction.
Suppose B emits photon, A receives it
Lorentz transformation of photon 4-momentum implies

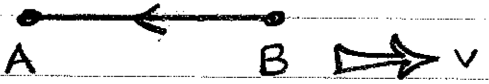
$$\underset{\nu' = \nu_{emit}}{h\nu'} = \underset{\nu = \nu_{obs}}{h\nu}\, \gamma\,(1 - n_x v)$$

So

$$\boxed{1+z = \frac{\nu_{emit}}{\nu_{obs}} = \gamma\,(1 - n_x v) = \gamma\,(1 - \mathbf{n}\cdot\mathbf{v})}$$

general Doppler shift formula

Case B receding from A



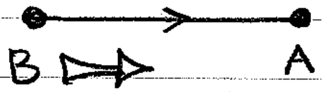
Then $n_x = -1$
(photon going in opposite direction than B).
whence

$$\boxed{1+z} = \gamma(1+v) = \frac{1+v}{\sqrt{1-v^2}} = \frac{1+v}{\sqrt{(1-v)(1+v)}}$$

$$\boxed{= \sqrt{\frac{1+v}{1-v}}}$$

receding Doppler shift formula

Case B approaching A



Then $n_x = 1$
(photon going in same direction as B).
whence

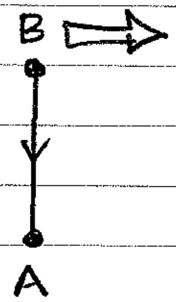
$$\boxed{1+z} = \gamma(1-v) = \frac{1-v}{\sqrt{1-v^2}} = \frac{1-v}{\sqrt{(1-v)(1+v)}}$$

$$\boxed{= \sqrt{\frac{1-v}{1+v}}}$$

approaching Doppler shift formula.

This is same as receding formula with $v \to -v$. So receding formula remains valid here too (with negative v).

Case B moving transverse to A

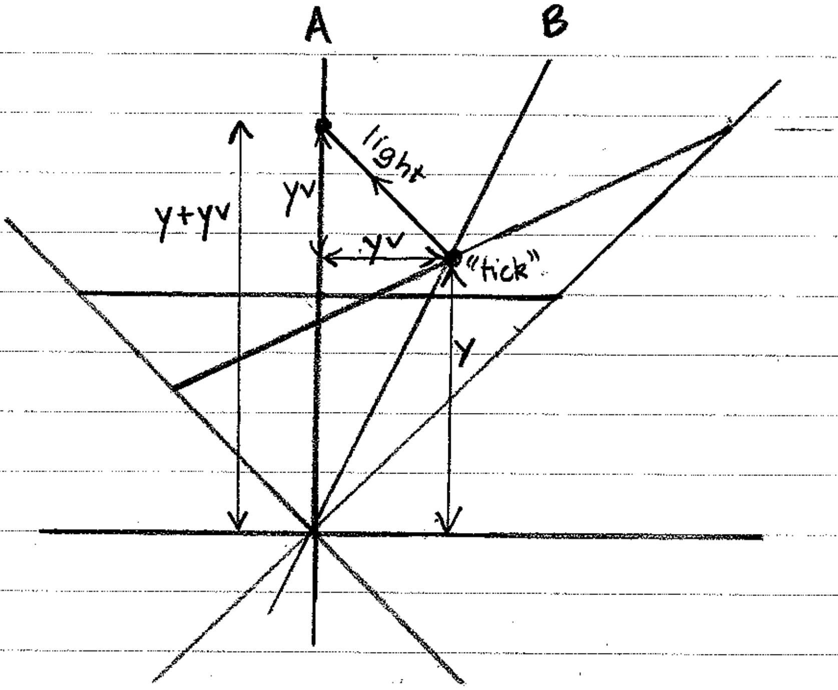


Then $n_x = 0$.

Whence

$$\boxed{1 + z = \gamma}$$ transverse Doppler shift formula

Apparent rate at which clocks tick



For "god-like" observer A
(who sees instantly – no travel time effects)
B's clock tick 1 time
while A's clock ticks y times

During this time B moves distance yv
so takes extra yv of light travel time
for A to see B tick.

So actually looks to A like

$$\frac{\text{Time passed on A clock}}{\text{Time passed on B clock}} = y + yv$$

$$= \sqrt{\frac{1+v}{1-v}}$$

Famous (receding) Doppler shift formula!

Velocity addition in special relativity

B moves relative to A at velocity v_1 in x-direction.

So
$$\underset{\substack{\uparrow\\ \text{coords of point}\\ \text{in A frame}}}{\begin{pmatrix} t \\ x \end{pmatrix}} = \begin{pmatrix} \gamma_1 & \gamma_1 v_1 \\ \gamma_1 v_1 & \gamma_1 \end{pmatrix} \underset{\substack{\uparrow\\ \text{coords of point}\\ \text{in B frame.}}}{\begin{pmatrix} t' \\ x' \end{pmatrix}}$$

C moves relative to B at velocity v_2 in x-direction.

So
$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma_2 & \gamma_2 v_2 \\ \gamma_2 v_2 & \gamma_2 \end{pmatrix} \underset{\substack{\uparrow\\ \text{coords of point}\\ \text{in C frame}}}{\begin{pmatrix} t'' \\ x'' \end{pmatrix}}$$

Hence
$$\begin{aligned}
\underset{\substack{\uparrow\\ \text{A-frame}}}{\begin{pmatrix} t \\ x \end{pmatrix}} &= \begin{pmatrix} \gamma_1 & \gamma_1 v_1 \\ \gamma_1 v_1 & \gamma_1 \end{pmatrix} \begin{pmatrix} \gamma_2 & \gamma_2 v_2 \\ \gamma_2 v_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} t'' \\ x'' \end{pmatrix} \\
&= \begin{pmatrix} \gamma_1 \gamma_2 (1 + v_1 v_2) & \gamma_1 \gamma_2 (v_1 + v_2) \\ \gamma_1 \gamma_2 (v_1 + v_2) & \gamma_1 \gamma_2 (1 + v_1 v_2) \end{pmatrix} \begin{pmatrix} t'' \\ x'' \end{pmatrix} \\
&= \begin{pmatrix} \Gamma & \Gamma V \\ \Gamma V & \Gamma \end{pmatrix} \underset{\substack{\nwarrow \text{ c-frame}}}{\begin{pmatrix} t'' \\ x'' \end{pmatrix}}
\end{aligned}$$

Thus C moves relative to A at velocity

$$\boxed{V = \frac{v_1 + v_2}{1 + v_1 v_2}}$$

is special relativistic formula for velocity addition.

Straightforward to confirm that
$$\Gamma = \frac{1}{\sqrt{1 - V^2}} = \gamma_1 \gamma_2 (1 + v_1 v_2)$$

Velocity addition in case where velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ are not aligned

Exercise:

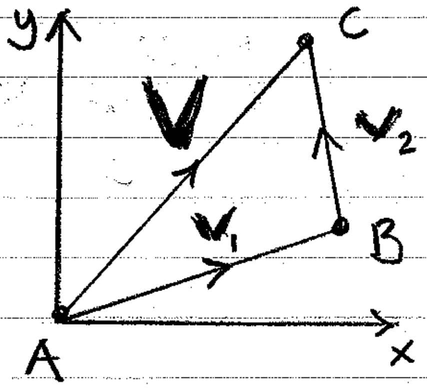
If $\mathbf{v}_1$ is along x-direction
$\quad$ ↖ B relative to A

but $\mathbf{v}_2$ is not necessarily along x-direction,
$\quad$ ↖ C relative to B

show that $\mathbf{V}$ is
$\quad$ ↖ C relative to A

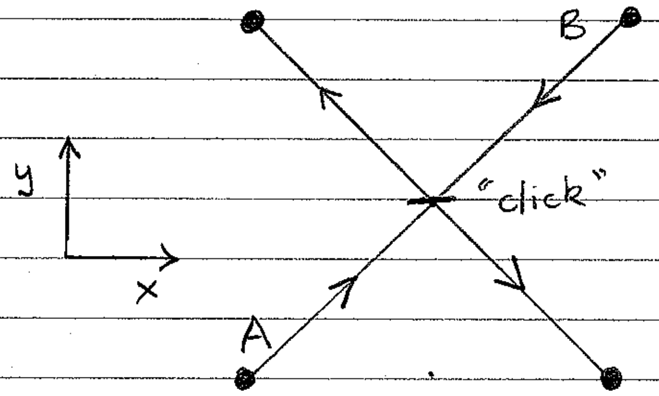
$$\boxed{V_x = \frac{v_{1x} + v_{2x}}{1 + v_{1x} v_{2x}}, \quad V_y = \frac{v_{2y}\sqrt{1-v_1^2}}{1 + v_{1x} v_{2x}}, \quad V_z = \frac{v_{2z}\sqrt{1-v_1^2}}{1 + v_{1x} v_{2x}}}$$

Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ appear asymmetrically. This reflects non-commutativity of velocity addition in special relativity.



Colliding billiard ball experiment

Consider 2 balls, equal mass m, colliding in following symmetrical fashion :



Center of mass frame

Before collision

$$\begin{pmatrix} v_{Ax} \\ v_{Ay} \end{pmatrix} = \begin{pmatrix} v \\ v \end{pmatrix}$$

$$\begin{pmatrix} v_{Bx} \\ v_{By} \end{pmatrix} = \begin{pmatrix} -v \\ -v \end{pmatrix}$$

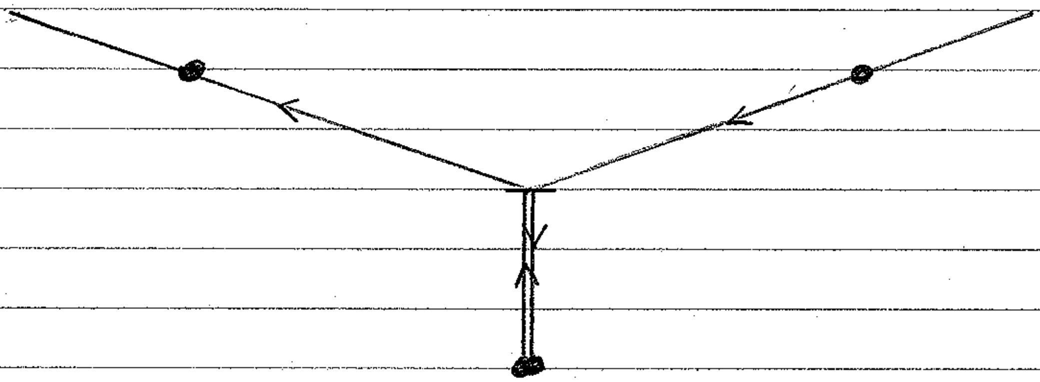
After collision

v_x remains same

$v_y \longrightarrow -v_y$

for both A & B.

Collision seen in primed frame moving at v in x-direction



Apply SR velocity addition rules.
Find

$$\begin{pmatrix} v'_{Ax} \\ \\ v'_{Ay} \end{pmatrix} = \begin{pmatrix} \dfrac{v_{Ax} - v}{1 - v v_{Ax}} \\ \\ \dfrac{v_{Ay}\sqrt{1-v^2}}{1 - v v_{Ax}} \end{pmatrix} = \begin{pmatrix} 0 \\ \\ \gamma v \end{pmatrix} \leftarrow \text{is faster than } v$$

$$\begin{pmatrix} v'_{Bx} \\ \\ v'_{By} \end{pmatrix} = \begin{pmatrix} \dfrac{v_{Bx} - v}{1 - v v_{Bx}} \\ \\ \dfrac{v_{By}\sqrt{1-v^2}}{1 - v v_{Bx}} \end{pmatrix} = \begin{pmatrix} \dfrac{-2v}{1+v^2} \\ \\ \dfrac{-v\sqrt{1-v^2}}{1+v^2} \end{pmatrix} \leftarrow \text{is slower than } v$$

What a mess!
Why? Because velocity is a 3D vector,
whose projection into 4D spacetime is yukky.

Colliding billiard balls using 4-vectors

Center of mass frame

$$p_A^i = \begin{pmatrix} \Gamma \\ \Gamma v \\ \Gamma v \\ 0 \end{pmatrix} \qquad V_A = V = \sqrt{2}\, v$$

$$\Gamma_A = \Gamma = \frac{1}{\sqrt{1-V^2}} = \frac{1}{\sqrt{1-2v^2}}$$

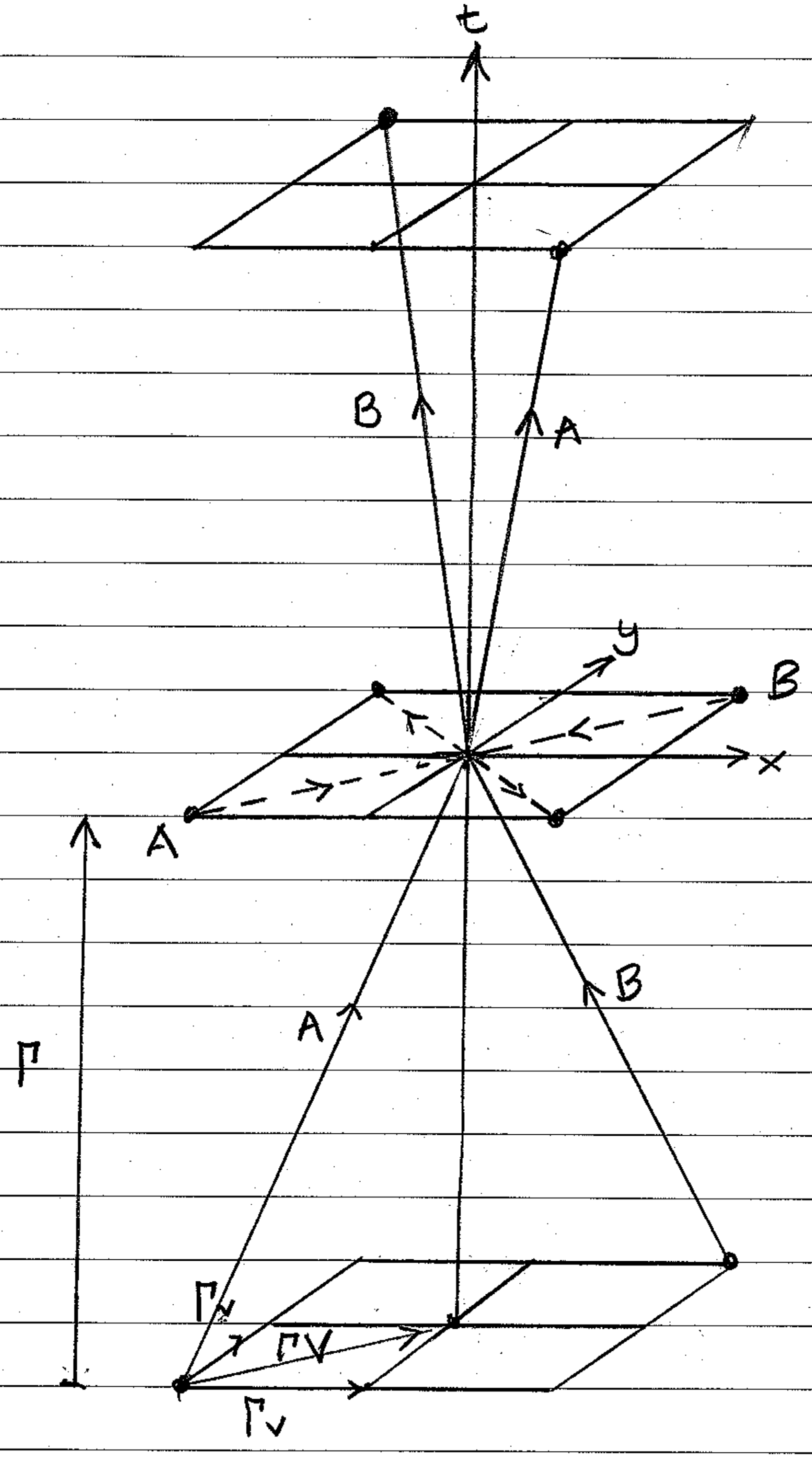
$$p_B^i = \begin{pmatrix} \Gamma \\ -\Gamma v \\ -\Gamma v \\ 0 \end{pmatrix}$$

Frame moving at v along x

$$p_A'^i = \underbrace{\begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}_{\text{Lorentz transf}} \underbrace{\begin{pmatrix} \Gamma \\ \Gamma v \\ \Gamma v \\ 0 \end{pmatrix}}_{p_A^i} = \frac{\Gamma}{\gamma} \begin{pmatrix} 1 \\ 0 \\ \gamma v \\ 0 \end{pmatrix}$$

$$p_B'^i = \quad '' \quad \underbrace{\begin{pmatrix} \Gamma \\ -\Gamma v \\ -\Gamma v \\ 0 \end{pmatrix}}_{p_B^i} = \gamma\Gamma(1+v^2) \begin{pmatrix} 1 \\ \frac{-2v}{1+v^2} \\ \frac{-v}{\gamma(1+v^2)} \\ 0 \end{pmatrix}$$

reproduces previously result much more easily.


t
B
A
y
B
x
A
A
B
Γ
Γv
ΓV
Γv

Statement of energy-momentum conservation

In any closed system, total energy-momentum is constant

$$\boxed{\sum_{\text{particles}} p^i = P^i = \text{constant}}$$

Review of SR

1. Philosophy ⟷ Physics

Only what is obseNed exists

- ⟷ e.g. absolute spacetime does not exist
- ⟷ relativity of obseNer & obseNee
- ⟷ symmetries
- ⟷ conseNation laws
- ⟷ forces.

2. Historical elements

p. 1.12.

3. Special Relativity

- Postulates
- Spacetime diagrams
- Simultaneity
- Lorentz transformation
- Scalar spacetime interval — proper time & distance.
- Time dilation — Fitzgerald-Lorentz contraction — velocity addition — paradoxes.

4. Mathematical apparatus

- Matrices
- Lorentz scalars, 4-vectors, tensors.

5. Energy - Momentum

- Energy - momentum 4-vector
- Lorentz covariance
- $E = mc^2$
- Rest mass.
- Energy - momentum conservation.

6. Photons

- Energy - momentum
- Redshift — Doppler shift
- Relativistic beaming

7. What do things actually look like near c?

- Velocity — superluminal motion
- Distortion of shapes
- Redshift ↔ rate at which clocks appear to tick.