

## Chapter 2

### Feshbach resonance formalism

#### 2.1 Feshbach resonance theory

We now briefly describe the Feshbach formalism and derive the elastic  $S$ - and  $T$ -matrices for two-body Feshbach resonant scattering. These matrices represent the transition probabilities for scattering between states within the resonant system. A more detailed and extended treatment of this formalism can be found in the literature [30, 31].

In Feshbach resonance theory, two projection operators  $P$  and  $Q$  are introduced which project onto the subspaces  $\mathcal{P}$  and  $\mathcal{Q}$  and satisfy the relations:

$$\begin{aligned} P &= P^\dagger & Q &= Q^\dagger \\ P^2 &= P & Q^2 &= Q \\ P + Q &= 1. \end{aligned} \tag{2.1}$$

These subspaces form two orthogonal components which together span the full Hilbert space of both scattering and bound wavefunctions  $|\psi\rangle = P|\psi\rangle + Q|\psi\rangle$ . The open and closed channels are contained in  $\mathcal{P}$  and  $\mathcal{Q}$ , respectively. The operators  $P$  and  $Q$  split the Schrödinger equation for the two-body problem  $(E - H)|\psi\rangle = 0$  into two parts:

$$(E - H_{PP})|\psi_P\rangle = H_{PQ}|\psi_Q\rangle, \tag{2.2}$$

$$(E - H_{QQ})|\psi_Q\rangle = H_{QP}|\psi_P\rangle, \tag{2.3}$$

where  $H_{PP} = PHP$ ,  $H_{PQ} = PHQ$ , etc., and  $\psi$  is the total scattering wavefunction. The projections on the two sub-spaces are indicated by  $P|\psi\rangle = |\psi_P\rangle$  and  $Q|\psi\rangle = |\psi_Q\rangle$ .

The Hamiltonian  $H = H^0 + V$  consists of the sum of single-particle interactions  $H^0$  and the two-body interaction  $V$ . We may formally solve Eq. (2.3) to find:

$$|\psi_Q\rangle = \frac{1}{E^+ - H_{QQ}} H_{QP} |\psi_P\rangle, \quad (2.4)$$

where  $E^+ = E + i\delta$ , with  $\delta$  approaching zero from positive values. Substituting this result into Eq. (2.2), the open channel equation can be written as

$$(E - H_{\text{eff}}) |\psi_P\rangle = 0, \quad (2.5)$$

where

$$H_{\text{eff}} = H_{PP} + H_{PQ} \frac{1}{E^+ - H_{QQ}} H_{QP}. \quad (2.6)$$

If we write Eq. (2.6) in the following form:

$$H_{\text{eff}} = H_{PP}^0 + V_{\text{eff}}, \quad (2.7)$$

we may identify an effective potential

$$V_{\text{eff}} = V_{PP} + H_{PQ} \frac{1}{E^+ - H_{QQ}} H_{QP}, \quad (2.8)$$

resulting from the coupling to the  $Q$  subspace. We have thus reduced the scattering problem to that of scattering off an effective potential composed of an additive combination of the open channel potential and a coupling to the closed channels. We will return to this result in our later discussion of the many-body physics of Feshbach resonant systems.

The trick to solving Eq. (2.5) is to expand the resolvent operator into the discrete and continuum eigenstates of  $H_{QQ}$ :

$$H_{\text{eff}} = H_{PP} + \sum_i \frac{H_{PQ} |\phi_i\rangle \langle \phi_i| H_{QP}}{E - \epsilon_i} + \int \frac{H_{PQ} |\phi(\epsilon)\rangle \langle \phi(\epsilon)| H_{QP}}{E^+ - \epsilon} d\epsilon. \quad (2.9)$$

Here the  $\epsilon_i$ 's and  $\epsilon$ 's are the uncoupled bound-state and continuum eigenvalues, respectively. In practice, only a few bound states will significantly affect the properties of

the open-channel. We will now assume that a small number of bound states dominate the problem and will neglect the continuum expansion in Eq. (2.9). Then the formal solution for  $|\psi_P\rangle$  is given by

$$|\psi_P\rangle = |\psi_P^+\rangle + \frac{1}{E^+ - H_{PP}} \sum_i \frac{H_{PQ}|\phi_i\rangle\langle\phi_i|H_{QP}|\psi_P\rangle}{E - \epsilon_i}, \quad (2.10)$$

where the scattered wavefunction  $|\psi_P^+\rangle$  is an eigenstate of the direct interaction  $H_{PP}$  (i.e.,  $(E^+ - H_{PP})|\psi_P^+\rangle = 0$ ) and satisfies outgoing, spherical wave boundary conditions. Likewise, we may define a scattered wavefunction  $|\psi_P^-\rangle$  satisfying incoming, spherical wave boundary conditions,  $(E^- - H_{PP})|\psi_P^-\rangle = 0$ . The scattered wavefunctions  $|\psi_P^\pm\rangle$  may be formally solved for with the result

$$|\psi_P^\pm\rangle = |\chi_P\rangle + \frac{V_{PP}}{E^\pm - H_{PP}}|\chi_P\rangle, \quad (2.11)$$

where the unscattered wavefunction  $|\chi_P\rangle$  is defined as an eigenstate of  $H_{PP}^0$ .

We now quantify the scattering behavior of a Feshbach resonance system by calculating the transition matrix (T-matrix). To begin, let us define a nonresonant T-matrix for scattering within the  $\mathcal{P}$  subspace as:

$$\mathcal{T}_P = \langle\chi_P|V_{PP} + V_{PP}\frac{1}{E^+ - H_{PP}}V_{PP}|\chi_P\rangle \quad (2.12)$$

$$= \langle\chi_P|V_{PP}|\psi_P^+\rangle. \quad (2.13)$$

In moving from Eq. (2.12) to Eq. (2.13) we have made use of Eq. (2.11).

To calculate the full scattering T-matrix  $\mathcal{T}$ , which accounts for the coupling to the  $\mathcal{Q}$  subspace, we must identify the effective potential as defined by Eq. (2.6). By multiplying Eq. (2.10) from the left with  $\langle\chi_P|V_{\text{eff}}$ , we derive the full, scattering T-matrix:

$$\mathcal{T} = \langle\chi_P|V_{\text{eff}}|\psi_P\rangle \quad (2.14)$$

$$= \mathcal{T}_P + \sum_i \frac{\langle\psi_P^-|H_{PQ}|\phi_i\rangle\langle\phi_i|H_{QP}|\psi_P\rangle}{E - \epsilon_i}, \quad (2.15)$$

where we have again made use of the relation between the unscattered state  $|\chi_P\rangle$  and the scattering wave-function  $|\psi_P^-\rangle$  as given in Eq. (2.11).

From the T-matrix, we can easily go to the  $S$ -matrix defined as:

$$\mathcal{S} = \langle \psi_P^- | \psi_P^+ \rangle. \quad (2.16)$$

For  $s$ -wave scattering, there exists a simple relation between the  $S$ - and T-matrix [32]:

$$\mathcal{S} = 1 - 2\pi i \mathcal{T}. \quad (2.17)$$

This allows us to rewrite Eq. (2.15) as

$$\mathcal{S} = \mathcal{S}_P - \sum_i \frac{2\pi i \langle \psi_P^- | H_{PQ} | \phi_i \rangle \langle \phi_i | H_{QP} | \psi_P \rangle}{E - \epsilon_i}, \quad (2.18)$$

where the non-resonant factor  $\mathcal{S}_P$  describes the direct scattering process within the subspace  $\mathcal{P}$ .

## 2.2 Single resonance

For the case of only one resonant bound state and only one open channel, Eq. (2.18) gives rise to the following elastic  $S$ -matrix element:

$$\mathcal{S} = \mathcal{S}_P \left[ 1 - \frac{2\pi i |\langle \psi_P^+ | H_{PQ} | \phi_1 \rangle|^2}{E - \epsilon_1 - \langle \phi_1 | H_{QP} \frac{1}{E^+ - H_{PP}} H_{PQ} | \phi_1 \rangle} \right]. \quad (2.19)$$

This relation is found by acting on both sides of Eq. (2.10) with  $\langle \phi_i | H_{QP}$  and substituting the result into Eq. (2.18). The direct, non-resonant  $S$ -matrix is related to the background scattering length  $a_{\text{bg}}$ , for  $k \ll a_{\text{bg}}^{-1}$ , by  $\mathcal{S}_P = \exp[-2ika_{\text{bg}}]$ . The term in the numerator gives rise to the energy-width of the resonance,  $\Gamma = 2\pi |\langle \psi_P^+ | H_{PQ} | \phi_1 \rangle|^2$ , which is proportional to the incoming wavenumber  $k$  and coupling constant  $\bar{g}_1$  [33]. The bracket in the denominator gives rise to a shift of the bound-state energy, and to an additional width term  $i\Gamma/2$ . If we denote the energy-shift between the collision continuum and the bound state by  $\bar{\nu}_1$ , and represent the kinetic energy simply by  $\hbar^2 k^2/m$ , the  $S$ -matrix element can be rewritten as

$$\mathcal{S}(k) = e^{-2ika_{\text{bg}}} \left[ 1 - \frac{2ik|\bar{g}_1|^2}{-\frac{4\pi\hbar^2}{m}(\bar{\nu}_1 - \frac{\hbar^2 k^2}{m}) + ik|\bar{g}_1|^2} \right]. \quad (2.20)$$

The resulting, total scattering length, coming from the relation

$$\lim_{k \rightarrow 0} T(k) = \frac{4\pi\hbar^2 a}{m}, \quad (2.21)$$

has a dispersive shape of the form

$$a = a_{\text{bg}} \left( 1 - \frac{m}{4\pi\hbar^2 a_{\text{bg}}} \frac{|\bar{g}_1|^2}{\bar{\nu}_1} \right), \quad (2.22)$$

which may directly be related to Eq. (1.2). The form of Eq. (2.22) allows us to extract the parameters of the resonance model from a plot of the scattering length versus magnetic field.

### 2.3 Double resonance

Often more than one resonance may need to be considered. For example, the scattering properties for the  $(f, m_f) = (1/2, -1/2)$  and  $(1/2, 1/2)$  channel of  ${}^6\text{Li}$  are dominated by a combination of two resonances: a triplet potential resonance and a Feshbach resonance. A mechanism of this sort may be inferred from Fig. 2.1 where the residual scattering length, which would arise in the absence of the Feshbach resonance coupling, would be very large and negative and vary with magnetic field. This should be compared with the value of the non-resonant background scattering length for the triplet potential of  ${}^6\text{Li}$  which is only  $31 a_0$ , which is an accurate measure of the characteristic range of this potential. An adequate scattering model for this system, therefore, requires inclusion of both bound-state resonances. The double-resonance  $S$ -matrix, with again only one open channel, follows then from Eq. (2.18) and includes a summation over two bound states. After solving for the  $\psi_P$  wave function, the  $S$ -matrix can be written as

$$S(k) = e^{-2ika_{\text{bg}}} \left[ 1 - \frac{2ik(|\bar{g}_1|^2\Delta_2 + |\bar{g}_2|^2\Delta_1)}{ik(|\bar{g}_1|^2\Delta_2 + |\bar{g}_2|^2\Delta_1) - \Delta_1\Delta_2} \right], \quad (2.23)$$

with  $\Delta_1 = (\bar{\nu}_1 - \hbar^2 k^2/m)4\pi\hbar^2/m$ , where  $\bar{\nu}_1$  and  $\bar{g}_1$  are the detuning and coupling strengths for state 1. Equivalent definitions are used for state 2. Later we will show that

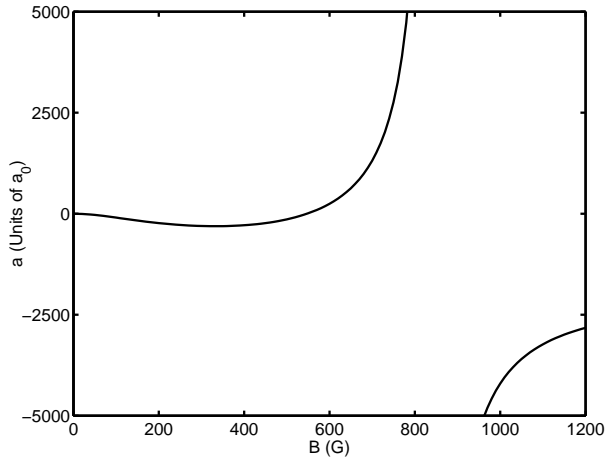


Figure 2.1: Scattering length as a function of magnetic field for the  $(f, m_f) = (1/2, -1/2)$  and  $(1/2, 1/2)$  mixed spin channel of  ${}^6\text{Li}$ .

this simple analytic Feshbach scattering model mimics the coupled channels calculation of  ${}^6\text{Li}$ .

Once again, the parameters of this model, which are related to the positions and widths of the last bound states, can be directly found from a plot of the scattering length versus magnetic field as given, for example, by Fig. 2.1. The scattering length behavior should be reproduced by the analytic expression for the scattering length following from Eq. (2.23):

$$a = a_{bg} - \frac{m}{4\pi\hbar^2} \left( \frac{|\bar{g}_1|^2}{\bar{\nu}_1} + \frac{|\bar{g}_2|^2}{\bar{\nu}_2} \right). \quad (2.24)$$

The advantage of a double-pole, over a single-pole,  $S$ -matrix parametrization is that we can account for the interplay between a potential resonance and a Feshbach resonance which, in principle, can radically change the scattering properties. This interplay is not only important for the description of  ${}^6\text{Li}$  interactions, but also for other atomic systems which have an almost resonant triplet potential, such as bosonic  ${}^{133}\text{Cs}$  [34, 35] and  ${}^{85}\text{Rb}$  [36].

In the many-body theories discussed in this thesis, the scattering properties are represented by a  $T$ -matrix instead of an  $S$ -matrix. We have shown for s-wave scattering

that there exists a simple relation between the two, however, the definition for  $T$  in the many-body theory will be slightly different in order to give it the conventional dimensions of energy per unit density:

$$\mathcal{T}(k) = \frac{2\pi\hbar^2 i}{mk} [\mathcal{S}(k) - 1]. \quad (2.25)$$

## 2.4 Coupled square-well scattering

We now describe the coupled-channels extension of a textbook single-channel square-well scattering problem [37]. Our motivation for studying this model is that we may take the limit of the potential range as  $R \rightarrow 0$ , resulting in an explicit representation of a set of coupled delta-function potentials. Such a set of potentials greatly simplifies the description in the many-body problem. The applicability of delta-function potentials, or contact potentials, to the many-body system is motivated by the fact that the length-scale associated with the range of the interatomic potential is, typically, much smaller than the length scale associated with the de Broglie wavelength, or other thermodynamic correlation scales. This means that we need not concern ourselves with the actual shape of the potential so long as the chosen potential reproduces the correct asymptotic scattering properties. We may, therefore, choose the simplest potential which satisfies the constraints of our problem.

The scattering equations for a system of coupled square wells may be written as

$$E_{\text{kin}}\psi^P(\mathbf{r}) = [H_0(\mathbf{r}) + V^P(\mathbf{r})]\psi^P(\mathbf{r}) + g(\mathbf{r})\psi^Q(\mathbf{r}), \quad (2.26)$$

$$E_{\text{kin}}\psi^Q(\mathbf{r}) = [H_0(\mathbf{r}) + V^Q(\mathbf{r}) + \epsilon]\psi^Q(\mathbf{r}) + g^*(\mathbf{r})\psi^P(\mathbf{r}), \quad (2.27)$$

with  $\epsilon$  the energy-shift of the closed channel and  $E_{\text{kin}} = \hbar^2 k^2/m$  the two-body kinetic energy. The coupled square well model encapsulates the general properties of two-body alkali interactions. In scattering events between alkalis, we can divide the internuclear separation into two regions: the inner region where the exchange interaction (the difference between the singlet and triplet potentials) is much larger than the hyperfine

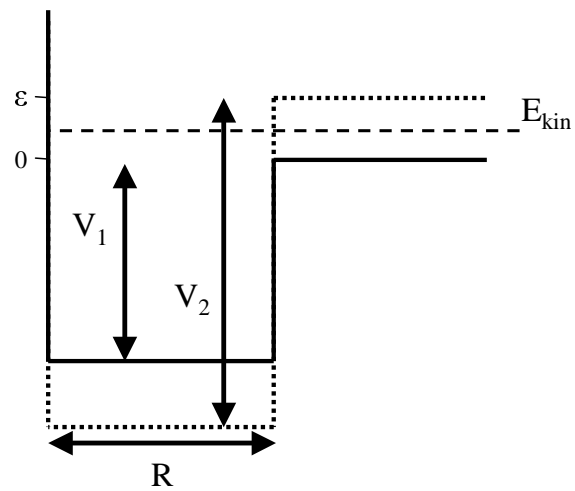


Figure 2.2: Illustration of the coupled square well system. The solid and dotted lines correspond to the molecular potentials  $V_1$  ( $P$ ) and  $V_2$  ( $Q$ ) inner  $r < R$  (outer  $r > R$ ) region, respectively. The dashed line corresponds to the kinetic energy  $E_{\text{kin}}$  in the open channel. The detuning  $\epsilon$  can be chosen such that a bound state of the square-well potential  $V_2$  enters the collision continuum causing a Feshbach resonance in the open channel.

splitting, and the outer region where the hyperfine interaction dominates. Here we make a similar distinction for the coupled square wells. In analogy to the real singlet and triplet potentials, we use, for the inner region, two artificial square-well potentials labelled as  $V_1$  and  $V_2$ . We take the coupling  $g(\mathbf{r})$  to be constant over the range of the potentials,  $r < R$ , and to be zero outside this range (see Fig. 2.2). For the outer region,  $r > R$ , the open and closed channel wavefunctions are given by  $u_P(r) \sim \sin k_P r$  and  $u_Q(r) \sim \exp(-k_Q r)$ , respectively, where  $u(r) = r\psi(r)$ . For the inner region  $r < R$  the wave functions are given by  $u_1(r) \sim \sin k_1 r$  and  $u_2(r) \sim \sin(k_2 r)$ . The relevant wavevectors are defined as:  $k_P = \sqrt{mE_{\text{kin}}}/\hbar$ ,  $k_Q = \sqrt{m(\epsilon - E_{\text{kin}})}/\hbar$ ,  $k_1 = \sqrt{m(E_{\text{kin}} + V_1)}/\hbar$ , and  $k_2 = \sqrt{m(E_{\text{kin}} + V_2 - \epsilon)}/\hbar$ .

This problem may now be solved by means of a basis rotation at the boundary  $R$  which gives rise to simple analytic expressions. The efficacy of this transformation may be understood by realizing that the effect of the potentials is to cause the wavefunction to accumulate a phase  $\phi_1 = k_1 R$  and  $\phi_2 = k_2 R$  at the boundary  $R$ . For  $r > R$ , we therefore consider one open channel and one closed channel with wavenumbers  $k_P(E_{\text{kin}})$  and  $k_Q(E_{\text{kin}}, \epsilon)$ .  $E_{\text{kin}}$  is the relative kinetic energy of the two colliding particles in the center of mass frame and  $\epsilon$  is the energy-difference of the two outer-range channels. In analogy with a real physical system, we can refer to the inner range channels as a molecular basis and the channel wave functions are just linear combinations of the  $u_1$  and  $u_2$  wave functions. The coupling strength is effectively given by the basis-rotation angle  $\theta$  for the scattering wave functions:

$$\begin{pmatrix} u_P(R) \\ u_Q(R) \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} u_1(R) \\ u_2(R) \end{pmatrix}, \quad (2.28)$$

allowing for an analytic solution of the scattering model. This leads to the following expression for the  $S$ -matrix:

$$\mathcal{S} = e^{-2ik_P R} [1 - (-2ik_P(k_2 \cot \phi_2 \cos^2 \theta + k_Q + k_1 \cot \phi_1 \sin^2 \theta)) / \dots] \quad (2.29)$$

$$(k_P k_Q + k_1 \cot \phi_1 (k_P \sin^2 \theta - k_Q \cos^2 \theta) + ik_2 \cot \phi_2 (k_1 \cot \phi_1 + k_P \cos^2 \theta + k_Q \sin^2 \theta)].$$

An extension to treat more than two coupled potentials, which would be required to model more than one resonance, is straightforward.

The parameters of the two wells have to be chosen such that the results of a real scattering calculation are reproduced. In fact, all the parameters are completely determined from the field dependence of the scattering length, and all other scattering properties can then be derived, such as the energy-dependence of the scattering phase shift. One way of selecting these parameters is to first choose a range  $R$ , typically of the order of an interatomic potential range ( $100 a_0$ ) or less. Now we have only to determine the set of parameters  $V_1$ ,  $V_2$ , and  $\theta$ . The potential depth  $V_1$  is chosen such that the scattering length is equal to the background scattering length  $a_{\text{bg}}$  while keeping  $\theta = 0$ . Also,  $V_1$  should be large enough that the wavenumber  $k_1$  depends only weakly on the scattering energy. Then we set  $\theta$  to be non-zero and change the detuning until a bound state crosses the collision threshold giving rise to a Feshbach resonance. The value of  $V_2$  is more or less arbitrary, but we typically choose it to be larger than  $V_1$ . Finally, we change the value of  $\theta$  to give the Feshbach resonance the desired width.

In Fig. 2.3 the coupled square-well system is compared with the Feshbach scattering theory for the scattering parameters of  $^{40}\text{K}$ . Despite the fact that there is a strong energy-dependence of the  $T$ -matrix, the two scattering representations agree very well. We will later show that the resulting scattering properties converge for  $R \rightarrow 0$ .

## 2.5 Comparison with coupled channels calculation

In the last section we argued that the double-well system is in good agreement with Feshbach scattering theory. Now we will show how well both the Feshbach theory and the  $R \rightarrow 0$  coupled square-well, or contact square-well, theory agree with the full coupled channels calculation [38]. In Fig. 2.4 we show the real and imaginary parts of the

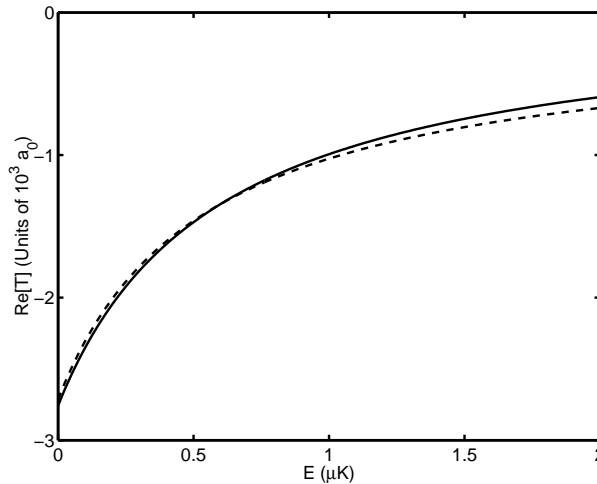


Figure 2.3: Comparison of coupled square-well scattering (solid line), with a potential range  $R = 1a_0$ , to Feshbach scattering (dashed line) for a detuning that yields a scattering length of about  $-2750 a_0$ . Similar agreement is found for all detunings.

$T$ -matrix applied to the case of  ${}^6\text{Li}$  and compare the contact square well and Feshbach scattering representations to a full coupled channels calculation. The agreement is surprisingly good and holds for all magnetic fields (i.e., similar agreement is found at all detunings).

In this section we have discovered the remarkable fact that even a complex system, including internal structure and resonances, can be described with contact potentials and a few coupling parameters. This was trivially known for off resonance scattering where only a single parameter, the scattering length, is required to encapsulate the collision physics. However, this has not been pointed out before for the resonance system where an analogous parameter set is required to describe a system with a scattering length that may pass through infinity!

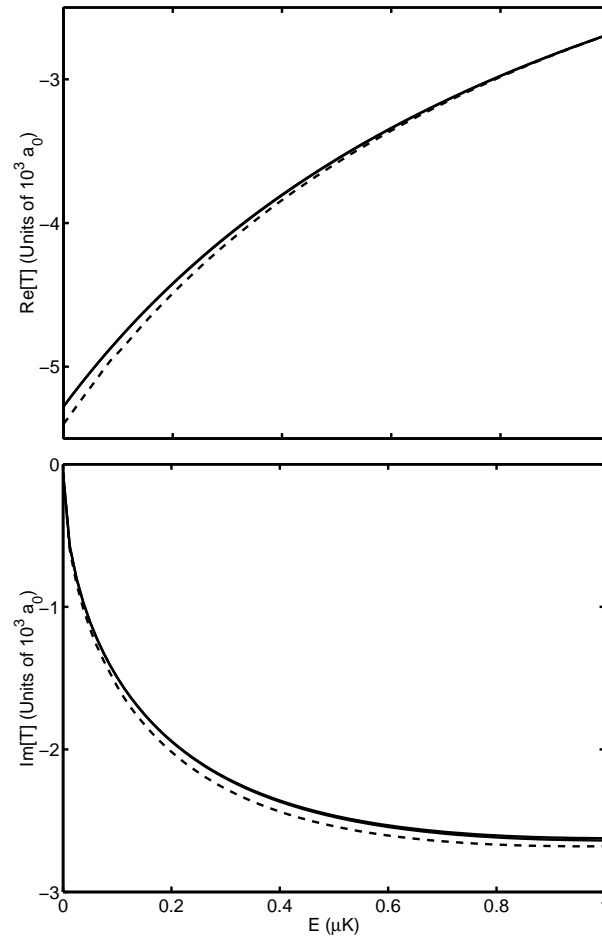


Figure 2.4: Real (top) and imaginary (bottom) part of the  $T$ -matrix, as a function of collision energy, for the Feshbach model and the contact square-well model (overlapping solid lines), and for a coupled channels calculation (dashed line). The atomic species considered is  ${}^6\text{Li}$ , for atoms colliding in the  $(f, m_f) = (1/2, -1/2)$  and  $(1/2, 1/2)$  channel near 800G.

## 2.6 Pseudo-potential scattering and renormalization

In forming a field theory to describe Feshbach resonant interactions we would prefer not to work with the complete interatomic potentials since they contain much more information than is actually needed and severely complicate the problem. Rather, we would like to work with a suitable set of pseudo-potentials that incorporate all the relevant scattering physics. As we have shown in the previous sections, a pair of coupled square wells may be accurately used to describe a Feshbach resonance. However, we had to adjust the various parameters, such as the depth and coupling of the wells, in order to produce the desired resonance. A correct field theoretic description of a resonant system must, similarly, adjust the pseudo-potentials. This adjustment procedure is termed “renormalization” and connects the pseudo-potential parameters in the field theory to the physical parameters describing the resonance. What’s more, if contact potentials are incorporated in the many-body theory, which is often done out of initial convenience, the renormalization works to remove divergences that will naturally arise due to the singularity of such a potential. To show how one properly renormalizes a Feshbach resonant field theory, we must first take a closer look at the underlying two-body physics.

We begin by solving the Lippmann-Schwinger scattering equation for a Feshbach resonance system. Our goal will be to derive explicit expressions which match the two-body scattering parameters describing the Feshbach resonance  $(\bar{U}, \bar{g}_i, \bar{\nu}_i)$  to the model potential parameters  $(U, g, \nu)$  entered into our field theory. The first step is to solve the scattering Eqs. (2.26) and (2.27) for these pseudo-potentials. As we have seen, we can formally solve the bound-state equations and make use of Eq. (2.9) to expand the Green’s function in bound-state solutions. In this case we may write

$$\psi^Q(\mathbf{r}) = \sum_i \frac{\phi_i^Q(\mathbf{r}) \int d^3r' \phi_i^{Q*}(\mathbf{r}') g^*(\mathbf{r}') \psi^P(\mathbf{r}')}{E^+ - \epsilon_i}, \quad (2.30)$$

with  $\phi_i^Q(\mathbf{r})$  a bound state wavevector and  $\epsilon_i$  its eigenenergy. For the purposes of this

section we will only consider a single resonance potential so will henceforth drop the  $i$  subscript. We now define an amplitude for the system to be in the bound state  $\phi = \langle \phi^Q | \psi^Q \rangle$ . This definition will later prove useful in the mean-field equations. Together with the open channel equation and the definition  $g(\mathbf{r}) = g(\mathbf{r})\phi^Q(\mathbf{r})$  we get a new set of scattering equations

$$\frac{\hbar^2 \mathbf{k}^2}{m} \psi(\mathbf{r}) = \left( -\frac{\hbar^2}{m} \nabla^2 + V(\mathbf{r}) \right) \psi(\mathbf{r}) + g(\mathbf{r})\phi \quad (2.31)$$

$$\frac{\hbar^2 \mathbf{k}^2}{m} \phi = \nu \phi + \int d^3 r' g(\mathbf{r}') \psi(\mathbf{r}'), \quad (2.32)$$

where we have dropped the  $P$  superscripts. A formal solution for  $\psi(\mathbf{r})$  may be given by the following relation:

$$\psi(\mathbf{r}) = \chi(\mathbf{r}) - \frac{m}{4\pi\hbar^2} \int d^3 r' \frac{e^{i|\mathbf{k}||\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|} [V(\mathbf{r}')\psi(\mathbf{r}') + g(\mathbf{r}')\phi], \quad (2.33)$$

where  $\chi(\mathbf{r})$  represents the unscattered component. The scattering amplitude  $f(\theta)$  is defined in the limit  $r \rightarrow \infty$  of the wavefunction by the relation

$$\psi(\mathbf{r}) = \chi(\mathbf{r}) + f(\theta) \frac{e^{ikr}}{r}. \quad (2.34)$$

By taking the  $r \rightarrow \infty$  limit of Eq. (2.33) we have an asymptotic form for the full scattered wavefunction

$$\psi(\mathbf{r}) = \chi(\mathbf{r}) - \frac{m}{4\pi\hbar^2} \frac{e^{ikr}}{r} \int d^3 r' e^{-i\mathbf{k}\cdot\mathbf{r}'} [V(\mathbf{r}')\psi(\mathbf{r}') + g(\mathbf{r}')\phi]. \quad (2.35)$$

A comparison of Eq. (2.35) to Eq. (2.34) gives the following form for the scattering amplitude:

$$f(\theta) = -\frac{m}{4\pi\hbar^2} \int d^3 r' e^{-i\mathbf{k}\cdot\mathbf{r}'} [V(\mathbf{r}')\psi(\mathbf{r}') + g(\mathbf{r}')\phi]. \quad (2.36)$$

In Fourier space this becomes

$$f(\mathbf{k}, \mathbf{k}') = -\frac{m}{4\pi\hbar^2} \left[ \int \frac{d^3 p}{(2\pi)^3} (V(\mathbf{k}' - \mathbf{p})\psi(\mathbf{p})) + g(\mathbf{k}')\phi \right]. \quad (2.37)$$

With the aid of the relationship  $\mathcal{T}(\mathbf{k}, \mathbf{p}) = -(4\pi\hbar^2/m)f(\mathbf{k}, \mathbf{p})$ , linking the T-matrix to the scattering amplitude, we may write the Fourier representation of the open channel wavefunction as

$$\psi(\mathbf{p}) = (2\pi)^3\delta(\mathbf{k} - \mathbf{p}) + \frac{\mathcal{T}(\mathbf{k}, \mathbf{p})}{\frac{\hbar^2\mathbf{k}^2}{m} - \frac{\hbar^2\mathbf{p}^2}{m} + I\delta}. \quad (2.38)$$

Substituting this result into Eq. (2.37) yields

$$\mathcal{T}(\mathbf{k}, \mathbf{k}') = V(\mathbf{k}' - \mathbf{k}) + \int \frac{d^3p}{(2\pi)^3} \frac{U(\mathbf{k}' - \mathbf{p})\mathcal{T}(\mathbf{k}, \mathbf{p})}{\frac{\hbar^2\mathbf{k}^2}{m} - \frac{\hbar^2\mathbf{p}^2}{m} + I\delta} + g(\mathbf{k}')\phi. \quad (2.39)$$

By rewriting the original closed channel scattering equation, Eq. (2.27), as

$$\phi = \frac{1}{\frac{\hbar^2\mathbf{k}^2}{m} - \nu} \int \frac{d^3p}{(2\pi)^3} g(\mathbf{p})\psi(\mathbf{p}), \quad (2.40)$$

we may eliminate  $\phi$  from Eq. (2.39). This results in the following integral equation for the off the energy-shell T-matrix

$$\mathcal{T}(\mathbf{k}, \mathbf{k}') = \left[ U(\mathbf{k}' - \mathbf{k}) + \frac{g(\mathbf{k}')g(\mathbf{k})}{\frac{\hbar^2\mathbf{k}^2}{m} - \nu} \right] + \int \frac{d^3p}{(2\pi)^3} \left[ \frac{\left( U(\mathbf{k}' - \mathbf{p}) + \frac{g(\mathbf{k}')g(\mathbf{p})}{\frac{\hbar^2\mathbf{k}^2}{m} - \nu} \right) \mathcal{T}(\mathbf{k}, \mathbf{p})}{\frac{\hbar^2\mathbf{k}^2}{m} - \frac{\hbar^2\mathbf{p}^2}{m} + I\delta} \right]. \quad (2.41)$$

We now have a general integral equation for the T-matrix. However, to make use of this relationship we will need to make some simplifying assumptions. The first being the separability of the potential  $U(k - k') = \lambda(k)\lambda(k')$ . The second, that the T-matrix is basically constant over the range of the integral allowing us to pull it outside of the integral. This is another way of saying that the T-matrix only contributes on the energy-shell. Third, we will assume s-wave scattering so the wavenumbers  $\mathbf{k} = \mathbf{k}'$ . This results in the algebraic equation for the T-matrix:

$$\mathcal{T}(\mathbf{k}) = \left[ \lambda(\mathbf{k})^2 + \frac{g(\mathbf{k})^2}{\frac{\hbar^2\mathbf{k}^2}{m} - \nu} \right] + \mathcal{T}(\mathbf{k}) \int \frac{d^3p}{(2\pi)^3} \left[ \frac{\left( \lambda(\mathbf{k})\lambda(\mathbf{p}) + \frac{g(\mathbf{k})g(\mathbf{p})}{\frac{\hbar^2\mathbf{k}^2}{m} - \nu} \right)}{\frac{\hbar^2\mathbf{k}^2}{m} - \frac{\hbar^2\mathbf{p}^2}{m} + I\delta} \right]. \quad (2.42)$$

The renormalization is performed by matching the potentials  $U, g$  and detuning  $\nu$  to the physical potentials  $\bar{U}, \bar{g}$  and the physical detuning  $\bar{\nu}$  as given by the form of the Feshbach theory (Eqs. (2.20) and (2.25)). Results for several types of model pseudo-potentials are given in Appendix. A.