Typically, we need to normalize a wavefunction so that it can be used as a probability
distribution. Normalize the following wavefunctions:
a) \( \sin(\frac{n\pi x}{L}) \) over the range \( 0 \leq x \leq L \)
b) The H 1s wavefunction: \( \exp(-r/a_o) \) in spherical (3D) coordinates. To
integrate in spherical coordinates, recall that the volume element is
\[ d\tau = r^2 dr \sin \theta \, d\theta \, d\phi \]
with the limits \( 0 \leq r \leq \infty \), \( 0 \leq \theta \leq \pi \), \( 0 \leq \phi \leq 2\pi \) (8 points)

Write the following expressions in Dirac Bra/Ket notation. Furthermore, assuming
\( \phi_1(x) \) and \( \phi_2(x) \) are normalized eigenfunctions of the Hamiltonian \( \hat{H} \), with total energies
\( E_1 \) and \( E_2 \), respectively, write the value of each expression.

a) \( \int \phi_1(x)\phi_2(x)dx \),  
b) \( \int \phi_2(x)\phi_2(x)dx \),  
c) \( \int \phi_2(x)\hat{H}\phi_2(x)dx \) (6 points)

Is the plane wave \( \psi = Ae^{-i\frac{px}{\hbar}-\omega t} \) a physically acceptable wavefunction? Why or why
not? (3 points)

The wavefunction for a particle confined to a one dimensional box is given in problem
1a. Assuming the length of the box is \( L=10 \) nm, use the normalized wavefunction to
calculate the probability of finding the particle in the region a) between 4.95 and 5.05
nm, b) between 1.95 and 2.05 nm, and c) in the right half of the box. (6 points)

Calculate the translational kinetic energy for the lowest energy state of an electron in
a one-dimensional box of length 1.06 Å. Give the result in J and eV. What wavelength
photon (in nm) would be required for the lowest energy transition of this electron? (4
points)

Let’s assume that a particle in a 1-D box of length \( a \) is in a state with wavefunction
\[ \psi(x) = N \left( 1 - \frac{4x^2}{a^2} \right) \quad \text{for} \quad -\frac{a}{2} \leq x \leq \frac{a}{2} \quad \text{and} \quad \psi(x) = 0 \quad \text{elsewhere}. \]
Also assume that we write
the particle in a box wavefunctions slightly differently than we did in class, so they are
defined on the same interval, centered on zero:
\[ \phi_n(x) = \sqrt{\frac{2}{a}} \cos \left( \frac{n\pi x}{a} \right) \quad (\text{for odd } n) \]
\[ \phi_n(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi x}{a} \right) \quad (\text{for even } n) \]

a) Plot or sketch the wavefunction \( \psi(x) \) within the box. (3 points)

b) Determine the value of the normalization factor \( N \). (4 points)
c) Is \( \psi(x) \) an eigenstate of the particle in a box Hamiltonian? Explain briefly. (2 points)

d) Is \( \phi_1(x) + \phi_2(x) \) an eigenstate of the particle in a box Hamiltonian? Explain briefly. (2 points)

e) Write the starting point for calculating the coefficients \( c_n \) for the expansion of \( \psi(x) \) in the basis of \( \phi_n(x) \). Determining the values of the coefficients involves messy algebra, but it is straightforward to argue that \( c_n = 0 \) either for odd or even values of \( n \). Which ones are zero? (3 points)

f) This expansion can be used to calculate the expectation value of the energy, \( \langle \hat{E} \rangle \) for \( \psi(x) \). The series expression only yields the exact value of \( \langle \hat{E} \rangle \) if an infinite number of terms is summed, but an approximate value can be obtained by considering only a few terms. The result for the nonzero coefficients from part e) is:

\[
c_n = \sqrt{15} \left( \frac{2}{n\pi} \right)^3 (-1)^{(n-1)/2}.
\]

Calculate the value of \( \langle \hat{E} \rangle \) in units of the ground state energy, if the series is truncated at \( n = 1 \), \( n = 3 \), and \( n = 5 \), respectively, with a precision of 4 significant figures. (3 points)

7. Consider a particle in a box with the eigenfunctions \( \phi_n(x) \). The system is in a state with the wave function \( \psi(x) = N[\phi_1(x) + 2\phi_2(x)] \).

a) Determine the normalization constant \( N \). (3 points)

b) Calculate the value of \( \langle \hat{E} \rangle \) for the normalized wave function in terms of \( E_1 \). (2 points)