1. Calculus
Consider the functions $g_1(x) = \sin \left( \frac{\pi x}{l} \right)$ and $g_2(x) = e^{-x^2} \cos \left( \frac{\pi x}{l} \right)$

a. calculate $\frac{dg_2(x)}{dx}$

b. calculate $\int_{-\ell/2}^{\ell/2} dx \ [g_1(x)]^2$

c. calculate $\int_{-\ell/2}^{\ell/2} dx \ g_1(x)$

Hint: Consider the symmetry of the integrand for a shortcut solution that does not involve explicitly solving the integral.

2. Complex numbers
Any complex number can be written as $z = a + ib$, where $a$ and $b$ are real numbers, and $i^2 = -1$. $a$ is called the “real part,” $\text{Re}(z)$ and $b$ is called the “imaginary part,” $\text{Im}(z)$ of $z$.

$z^* = (a+ib)^* = a - ib$ is called the “complex conjugate” of $z$. We can visualize complex numbers as points in the complex plane.

a. Prove that $z^*z$ is always real and non-negative.

b. Provide a graphical interpretation of the value of $z^*z$ in terms of the complex plane.

c. For $d = \frac{-1+i}{1-2i}$ calculate $\text{Re}(d)$ and $\text{Im}(d)$.

d. The relation $e^{i\phi} = \cos \phi + i \sin \phi$ is called Euler’s formula. Any complex number $z$ can be written in the “polar form” as $z = Ae^{-i\theta}$ which has a simple geometric meaning (see graphic below).
Show that $\sqrt{-1} = \frac{1}{\sqrt{2}} (1 - i)$ using only geometry, the polar form, Euler’s formula and trigonometry. (3 points)


a. Find a solution of the differential equation $\frac{\partial f(x)}{\partial x} - \alpha f(x) = 0$ obeying the boundary condition $f(x_0) = A$ (4 points)

Hint: Use one (and only one!) exponential function as a trial function.

b. Qualitatively sketch $\text{Re}[f(x)]$ from (3a) for negative real values of $\alpha$. Construct a similar graph for $\alpha$ imaginary, with $\text{Im}(\alpha) = a > 0$ (assume that $A$, $x$, $x_0$ are positive real numbers). Please draw the sketches by hand, i.e. think for yourself, not relying on your graphing calculator! Also, mark the salient features of the graph, such as values at special points, etc. (4 points)