NAME: 

Directions: Complete all problems (total of 60 pts). To receive full credit for a problem, you must use the correct units and correct number of significant figures, and neatly show your work. If the writing is illegible, it will not be graded.

\[ h = 6.626 \times 10^{-34} \text{ J s} \]
\[ k_B = 1.381 \times 10^{-23} \text{ J K}^{-1} \]
\[ 1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} \]
\[ 1 \text{ nm} = 10^{-9} \text{ m} \]
\[ 1 \text{ W} = 1 \text{ J/sec} \]
\[ 1 \text{ J} = 1 \text{ kg m}^2 \text{ s}^{-2} \]
\[ 1 \text{ N} = 1 \text{ kg m s}^{-2} \]
\[ h = \frac{\hbar}{2\pi} \]
\[ c = 2.998 \times 10^{10} \text{ cm/sec} \]
\[ u_{\text{atomic mass}} = 1.661 \times 10^{-27} \text{ kg} \]
\[ 1 \text{ eV} = 1.602 \times 10^{-19} \text{ J} \]
\[ m_e = 9.109 \times 10^{-31} \text{ kg} \]
\[ c = \lambda \nu \]
\[ E = h\nu \]
\[ \Delta p \Delta q \geq \frac{\hbar}{4\pi} \]
\[ \lambda = \frac{h}{p} \]
\[ E_{\text{kinetic}} = \frac{1}{2}mv^2 \]
\[ p = mv \]

Integrals:
\[ \int_0^\infty e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{2\sqrt{a}} \quad \text{for } a > 0 \]
\[ \int_0^\infty x^2 e^{-ax^2} \, dx = \frac{\sqrt{\pi}}{4\sqrt{a^3}} \quad \text{for } a > 0 \]
\[ \int_0^\infty x^{2n+1} e^{-ax^2} \, dx = \frac{n!}{2a^{n+1}} \quad \text{for } a > 0, \ n = 0, 1, 2, ... \]
1. (5 pts) Lasers have been proposed as a means for propelling spacecraft in interstellar space. Let's consider the feasibility of this idea with a small satellite. If a photon-powered spacecraft of mass 10.0 kg emits radiation of wavelength 1000 nm, with a 3 kW laser, entirely in the backward direction, to what speed will it have accelerated after 10.0 years in free-space? EACH PHOTON INCREASES CRAFT MOMENTUM BY $h/\lambda$

FINAL CRAFT MOMENTUM $P = Nh/\lambda$ FOR $N$ PHOTONS

FINAL SPEED WILL BE $N h/\lambda M$. RATE OF PHOTON EMISSION IS

POWER / (PHOTON ENERGY) SO $N = \frac{P}{h c} \times \frac{h}{\lambda M}$

$V = \frac{P}{c m}$

$V = \frac{(10 \text{ YRS})(365 \text{ DAYS/YR})(24 \text{ HRS/DAY})(3600 \text{ SEC/HR})(3 \times 10^3 \text{ W})}{(2.938 \times 10^8 \text{ M/SEC})(10 \text{ KG})} = 3.16 \text{ M/SEC}$

2. (5 pts) The work function for metallic rubidium is 2.09 eV. Calculate the kinetic energy and the speed of the electrons emitted by light of wavelength (a) 650 nm, (b) 195 nm.

(a) $E_{\text{photon}} = \hbar c/\lambda$ so $E_k = \hbar c/\lambda - E_\Phi = \frac{1}{2} m E_k^2$ so $V = \sqrt{\frac{2E_k}{m_e}}$

(b) $E_k = \left[\frac{6.626 \times 10^{-34} \text{ J s}}{1.6 \times 10^{-19} \text{ eV}}\right] \left(\frac{2.938 \times 10^8 \text{ M/s}}{650 \times 10^{-9} \text{ m}}\right) - (2.09 \text{ eV})(1.6 \times 10^{-19} \text{ eV})$

$E_k < 0 \rightarrow$ NO PHOTON EMISSION BECAUSE $h \lambda < \Phi$

(c) $E_k = \left[\frac{6.626 \times 10^{-34} \text{ J s}}{1.6 \times 10^{-19} \text{ eV}}\right] \left(\frac{2.938 \times 10^8 \text{ M/s}}{195 \times 10^{-9} \text{ m}}\right) - (2.09 \text{ eV})(1.6 \times 10^{-19} \text{ eV})$

$E_k = 6.84 \times 10^{-19} \text{ J}$, $V = \sqrt{\frac{2(6.84 \times 10^{-19} \text{ J})}{3.11 \times 10^{-3} \text{ kg}}} = 1.23 \times 10^6 \text{ M/SEC}$

3. (4 pts) To a good approximation, an electron in a semiconductor quantum dot can be treated as a particle in a 3-D cubic box. If the emission of red light (650 nm) corresponds to a quantum transition from $n_1 = n_2 = n_3 = 2$ to $n_1 = n_2 = n_3 = 1$, what is the size of the quantum dot?

CUBIC BOX, SO $a = b = c \rightarrow E = \left(\frac{1}{8}\hbar m a^2\right)(m_1^2 + m_2^2 + m_3^2)$

$E_{222} = \frac{12\hbar^2}{8\hbar m a^2}$

$E_{111} = \frac{3\hbar^2}{8\hbar m a^2}$

$AE = \frac{9\hbar^2}{8\hbar m a^2} = \frac{hc}{\lambda}$

So $a = \left(\frac{9\hbar \lambda}{8\hbar m c}\right)^{1/2}$

$= \left[\frac{9 \times 6.626 \times 10^{-34} \text{ J s}}{8 \times 3.11 \times 10^{-3} \text{ kg}}(650 \times 10^{-9} \text{ m})\right]^{1/2}$

$a = 1.33 \times 10^{-9} \text{ M}$

OK TO GIVE VOLUME INSTEAD OF $a$

$V = 2.35 \times 10^{-27} \text{ M}^3$
4. (4 pts) Calculate the probability that a particle in a box of length $L$ will be found between 0.65$L$ and 0.67$L$, assuming it is (a) in the ground state, (b) in the first excited state. To simplify the calculation, assume the wavefunction is constant in this range.

(a) $n=1; \quad P = \frac{2}{L} \int_{0.65L}^{0.67L} \psi^* \psi \, dx = \frac{2(0.02L)}{L} \sin^2 \left( \frac{1}{2} \pi \frac{x}{L} \right) dx = \frac{2 \Delta x}{L} \sin^2 \left( \frac{1}{2} \pi \frac{x}{L} \right) \quad \text{where} \quad \Delta x = 0.02L, \quad x = 0.66L$ (center of interval)

(b) $n=2; \quad P = \frac{2}{L} \int_{0.65L}^{0.67L} \psi^* \psi \, dx = \frac{2(0.02L)}{L} \sin^2 \left( \frac{2}{2} \pi \frac{x}{L} \right) dx = 0.029$

5. (6 pts) Calculate the expectation values of $p$ and $p^2$ for the ground state of a particle in a box.

$$\psi_i = \sqrt{\frac{2}{L}} \sin \left( \frac{\pi x}{L} \right) \quad p = -i \frac{\partial}{\partial x}$$

$$\langle p \rangle = \langle \psi_i | p | \psi_i \rangle = \int_0^L \psi_i^* p \psi_i \, dx = -i \frac{\partial}{\partial x} \int_0^L \psi_i^2 \sin \left( \frac{\pi x}{L} \right) \sin \left( \frac{\pi x}{L} \right) \, dx$$

OR $\langle p \rangle = 0$ because of equal likelihood of $+p$ and $-p$ momenta.

$\langle p^2 \rangle = \langle \psi_i | p^2 | \psi_i \rangle = \langle \psi_i | 2mE_{kin} | \psi_i \rangle = 2mE_1, \quad \langle \psi_i | \psi_i \rangle = 2mE_1$

$$E_{kin} = E_1 = \frac{p^2}{2m} = \frac{L^2}{8mL^2} \quad \text{so} \quad \langle p^2 \rangle = \frac{p^2}{4L^2}$$

6. (6 pts) What is the implication for the simultaneous measurement of two observables if their corresponding operators have a set of eigenfunctions in common? Demonstrate your point by using operator algebra.

They commute = no uncertainty in simultaneously measuring $\hat{A}$ and $\hat{B}$

$$\hat{A} | \phi_i \rangle = a_i | \phi_i \rangle \quad \hat{B} | \phi_i \rangle = b_i | \phi_i \rangle$$

$\hat{A} \hat{B} | \phi_i \rangle = \hat{A} (b_i | \phi_i \rangle) = a_i b_i | \phi_i \rangle$

$\hat{B} \hat{A} | \phi_i \rangle = \hat{B} (a_i | \phi_i \rangle) = b_i a_i | \phi_i \rangle$

Order of measurements doesn’t matter. Observables are compatible.

$$\hat{A} \hat{B} | \phi_i \rangle = a_i b_i | \phi_i \rangle = \hat{B} \hat{A} | \phi_i \rangle$$
7. (10 pts) Consider the 1-D wavefunction (for which the real part of \(\Psi(x)\) is shown in the bottom frame) obtained by solving the Schrödinger equation for a particle moving on a step-function potential which increases from 0.000 to 0.500 eV at \(x=0\). The total energy of the particle is 0.710 eV.

\[
\begin{align*}
\text{Total energy} \\
\text{Potential energy}
\end{align*}
\]

\[
\begin{align*}
\text{Wave Function}
\end{align*}
\]

\(\text{(nm)}\)

a. Is the momentum of the particle larger in the region \(x<0\) or \(x>0\)? In only one sentence, explain why.

\[\text{LARGER IN } \underline{x<0} \text{ BECAUSE } \lambda \text{ IS SHORTER} \quad \text{AND } P = \frac{h}{\lambda} = mV\]

b. Given that the wavelength in the region \(x<0\) is 1.50 nm, calculate the mass of the particle.

\[P = \frac{h}{\lambda} = \frac{\left(6.626 \times 10^{-34} \text{ J s}\right)}{1.5 \times 10^{-9} \text{ m}} = 4.42 \times 10^{-25} \text{ kg m/s} \quad \text{[4 pts]}
\]

\[E = (0.710 \text{ eV})(1.6 \times 10^{-19} \text{ J/eV}) = 1.136 \times 10^{-19} \text{ J} \quad \text{[4 pts]}
\]

\[E_k = \frac{P^2}{2m} \quad \text{so} \quad m = \frac{P^2}{2E_k} = \frac{(4.42 \times 10^{-25} \text{ kg m/s})^2}{2(1.136 \times 10^{-19} \text{ J})} = 6.59 \times 10^{-5} \text{ kg} \quad \text{[4 pts]}
\]

c. Calculate the classical velocity of the particle in the region \(x>0\).

\[\text{USE MASS FROM } (b) \text{ AT } E_{\text{total}} - E_{\text{potential}} = 0.710 - 0.500 \text{ eV} \quad \text{[2 pts]}
\]

\[E = 0.210 \text{ eV} \quad \text{[2 pts]}
\]

\[V = \sqrt{\frac{2E}{m}} = \sqrt{\frac{2 \times 3.364 \times 10^{-20} \text{ J}}{6.59 \times 10^{-5} \text{ kg}}} = 2.79 \times 10^5 \text{ m/s} \quad \text{[2 pts]}
\]
8. (8 pts) Write the following expressions in Dirac Bra/Ket notation. Furthermore, assuming \( \phi_1(x) \) and \( \phi_2(x) \) are normalized eigenfunctions of the Hamiltonian \( \hat{H} \), with total energies \( E_1 \) and \( E_2 \), and kinetic energies \( E_{\text{kin}1} \) and \( E_{\text{kin}2} \), respectively, write the value of each expression.

\[
\int \phi_1^*(x) \phi_2(x) \, dx = \langle \phi_1 | \phi_2 \rangle = 0
\]

\[
\int \phi_1^*(x) \phi_2(x) \, dx = \langle \phi_2 | \phi_2 \rangle = 1.
\]

\[
\int \phi_1^*(x) \hat{p}^2 \phi_1(x) \, dx = \langle \phi_1 | \hat{p}^2 | \phi_1 \rangle = \langle \phi_1 | 2m \hat{E}_{\text{kin}1} | \phi_1 \rangle = 2m \hat{E}_{\text{kin}1}
\]

\[
\int \phi_2^*(x) \hat{H} \phi_2(x) \, dx = \langle \phi_2 | \hat{H} | \phi_2 \rangle = E_2 \langle \phi_2 | \phi_2 \rangle = E_2
\]

9. (12 pts) The state of a particle is given by \( \psi(x) = (2\pi/a)^{1/4} e^{-ax^2} \), \(-\infty < x < \infty \), where \( a \) is a constant. Verify that this state is consistent with the Heisenberg uncertainty principle.

\[
\Delta p \Delta x \geq \frac{\hbar}{2} \quad \text{and} \quad \Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \quad \Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}
\]

\[
\langle x \rangle = \int x |\psi|^2 \, dx = \int_{-\infty}^{\infty} \frac{(2\pi)^{1/4} a}{\pi} e^{-ax^2} \left( \frac{a}{\pi} \right)^{1/4} e^{-ax^2} \, dx = \left( \frac{2\pi}{a^2} \right)^{1/4} \frac{\pi^{1/4}}{2(2a)^{3/4}} = \frac{\pi^{1/4}}{2(2a)^{3/4}} \quad \text{ODD FUNCTION!}
\]

\[
\langle x^2 \rangle = \int_{-\infty}^{\infty} \left( \frac{2\pi}{a} \right)^{1/2} e^{-ax^2} x^2 e^{-ax^2} \, dx = \left( \frac{2\pi}{a^2} \right)^{1/2} \frac{\pi^{1/2}}{2(2a)^{3/2}} = \left( \frac{\pi}{2} \right)^{1/4} \left( \frac{2a}{\pi} \right)^{3/4}
\]

\[
\Rightarrow \Delta x = \frac{\pi^{1/4}}{2(2a)^{3/4}} \quad \Rightarrow \Delta p = \sqrt{\frac{\hbar}{2}}
\]

\[
|\langle p \rangle| = \int \left( \frac{2\pi}{a} \right)^{1/2} e^{-ax^2} \left( ax^2 - \frac{a}{2} \right) \, dx = \int \left( \frac{2\pi}{a} \right)^{1/2} e^{-ax^2} \left( ax^2 - \frac{a}{2} \right) \, dx = 0
\]

\[
|\langle p^2 \rangle| = \int \left( \frac{2\pi}{a} \right)^{1/2} e^{-ax^2} \left( 4ax^2 - 2a \right) \, dx = \int \left( \frac{2\pi}{a} \right)^{1/2} e^{-ax^2} \left( 4ax^2 - 2a \right) \, dx = \left( \frac{2\pi}{a} \right)^{1/2} \frac{\pi^{1/2} a}{2(2a)^{3/2}} = \left( \frac{\pi}{2} \right)^{1/4} \left( \frac{2a}{\pi} \right)^{3/4}
\]

\[
\Rightarrow \Delta p = \sqrt{\frac{\hbar}{2}}
\]

\[
\Rightarrow \sqrt{\Delta x} = \frac{
}{(2\pi/a)^{1/4} \sqrt{a} \sqrt{\Delta x}} = \frac{|\langle p \rangle|}{\Delta p} = \frac{\pi^{1/4}}{2(2a)^{3/4}} \sqrt{\frac{\hbar}{2}}
\]

\[
\text{MINIMUM UNCERTAINTY WAVEPACKET}
\]