Pulse shape

What’s your pulse shape

- Gaussian
- Hyperbolic secant

This is an interesting question

Caveat:
Must be in the regime where dominant pulse shaping mechanism is consistent with the theory
Describing the pulse shaping dynamics

“Continuous” Approaches:

Derive a differential equation describing the evolution of the envelope

1) Nonlinear Schrödinger Equation
   Includes Kerr nonlinearity (self phase modulation) and dispersion
   Solution is locally invariant \(\rightarrow\) pulse that does not change with propagation at all

2) “Master” Equation (Haus)
   Includes saturable absorber & gain, spectral filtering (extensions include dispersion and nonlinearity)
   Solution that is self-consistent after one round trip
   Ordering of elements does not matter

Both of these give \(\text{sech}^2(t)\) solutions

Both consider only the net dispersion in cavity, ignore that it is due to compensation between regions of opposite sign

“Dispersion managed” approaches:

Explicitly include variation regions of opposite dispersion.

Gives Gaussian solution in limit that changes in dispersion are large compared to net
Nonlinear Schrödinger Equation

Recall propagation equation from 2\textsuperscript{nd} lecture:

\[
\frac{\partial}{\partial z'} \hat{E} - \frac{i}{2} k''_0 \frac{\partial^2}{\partial t'^2} \hat{E} = 0
\]

We need to add nonlinearity, given by 
\[i \gamma |\hat{E}|^2 \hat{E}\] where \[\gamma = \frac{n_2 \alpha_0}{c A_{eff}}\]
for a beam with an “effective” cross-section area of \(A_{eff}\) yielding

\[
i \frac{\partial}{\partial z'} \hat{E} = \frac{1}{2} k''_0 \frac{\partial^2}{\partial t'^2} \hat{E} + \gamma |\hat{E}|^2 \hat{E}
\]

NLSE

Assume a solution

\[
\hat{E}(z', t') = \sqrt{P_0 \text{sech} \left( \frac{t'}{t_0} \right)} e^{iyP_0 z'}
\]

“soliton” with peak power \(P_0\) and FWHM 1.763 \(t_0\)

Substitution into NLSE yields relationship

\[
\frac{\gamma P_0 t_0}{\beta_2} = 1
\]

This describes a family of solutions, typically this determines \(t_0\) as other parameters are fixed by operating conditions
Solitons I

Where’d the $\text{sech}(t)$ come from? Just a lucky guess?

Nope, it can be derived from “inverse scattering theory”

Developed by Zakharov and Shabat

Classic text: Ablowitz and Segur “Solitons and the inverse scattering transform” (1981)

Inverse scattering theory yields

A set of solutions, corresponding to “eigenvalues” $N$

The $N=1$ solution is $\text{sech}(t)$

Higher order solutions are more complicated and periodic

The pulse velocity depends on $N$

Plus a “radiation” field

(analogy to quantum mechanics: think of eigenstates and radiation field like the bound and continuum wavefunctions of a finite potential well respectively)

An arbitrary pulse is decomposed into eigenstates + radiation

Eigenstates propagate unchanged
Decomposition has an interesting result: an arbitrary initial condition evolves into a soliton:

- Decompose into eigenstate + radiation $\rightarrow$ radiation spreads and dissipates, leaving soliton
- Implies stability/robustness against perturbation

Soliton theory seems like a pretty major simplification

- Ignored “dissipative” terms
  - Gain & Loss
    - Saturation of Gain & Loss (remember gain is always saturated in a laser!)
    - Higher order dispersion
- Ignored “lumped” nature of elements in a bulk optic laser

Soliton theory does a pretty good job when these are reasonable, e.g., for a fiber laser where

- Pulse shaping is dominated by GVD and nonlinearity
  - Which are distributed throughout the cavity

Tends to underestimate stability regimes (due to no saturation)

Nevertheless, it does give insight into one phenomena that occurs in bulk optic (KLM Ti:sapphire) lasers…
Solitons: Resonant ("Kelley") Sidebands

Look closely at Ti:sapphire spectrum:

Often notice a pair of “sidebands”

Usually requires log scale

Data at right for laser where they are particularly strong

Due to “phase matching” of radiation field arising from perturbations by gain/loss

1) Assume you have a proper soliton
2) It hits output coupler, pulse energy decreases
3) It has to get longer in time/narrower in spectrum to still be soliton
4) Excess spectrum is “shed” – travels at different speed
5) Due to periodicity of cavity, radiation of certain wavelengths add constructively $\Rightarrow$ phase matching of side bands

\[
\frac{\gamma P_0 t_0}{\beta_2} = 1
\]

Strength of radiation field $\leftrightarrow$ magnitude of perturbation

Spacing of sidebands $\leftrightarrow$ inverse cavity length

Note that the sidebands can “clamp” the spectral width, limits minimum duration
The NLSE is “conservative” as it does not have any terms describing loss (or gain).

Including these “dissipative” terms, and their saturation, results in the Ginzburg-Landau Equation:

$$\frac{\partial U}{\partial z} = \gamma U + \left(\frac{1}{\Omega} - i \frac{D}{2}\right) \frac{\partial^2 U}{\partial t^2} + \left|U\right|^2 U + \delta \left|U\right|^4 U$$

A pulse solution is

$$U[t, z] = \frac{A}{\sqrt{\cosh(t/\tau) + B}} e^{-i \beta / 2 \ln[\cosh(t/\tau) + B] + i \theta z}$$

Fiber laser experiments:

(pump power is control knob)

Write an “operator” for action of each element in the cavity on the pulse envelope.

In steady state, the net action of all operators has to have no effect:

\[ [g - l + S + D + i\psi + T_D + \gamma]E(t) = 0 \]

- **Nonlinearity**
- **Time delay**
- **Phase shift**
- **Dispersion**
- **Spectral filtering**
- **Loss**
- **Gain**

Needed for self-consistency because other elements can cause delay or phase shifts.

Can be combined (dispersion = imaginary spectral filter).

May be time dependent, Frequency dependence included in \( S \).

What are each of these operators?

Easy ones:

- \( \psi \) – just a number

\[ T_D = t_D \frac{d}{dt} \]
Spectral filter, gain, loss, and dispersion

Gain $g$ with a bandwidth $\omega_g$, in frequency domain transforms $E(\omega)$ to

$$\hat{E}'(\omega) = \left[1 + g \left(1 - \frac{(\omega - \omega_0)^2}{\omega_g^2}\right)\right] \hat{E}(\omega)$$

Where we have assumed the filter function is parabolic with frequency

Fourier transform in to time domain

$$\hat{E}'(t) = \left[1 + g \left(1 + \frac{1}{\omega_g^2} \frac{d^2}{dt^2}\right)\right] \hat{E}(t)$$

Loss is same, with $g \to -l$, typically lump together into single spectral filter

Non saturable loss and gain are easy, just numbers $g_0, l_0$

Group velocity dispersion is similar, just a complex spectral filter

$$\hat{E}'(t) = \left[1 + ik'_{0} \frac{d^2}{dt^2}\right] \hat{E}(t)$$
Active modelocker

\[
\left[ g \left( 1 + \frac{1}{\omega_g^2} \frac{d^2}{dt^2} \right) - l - m(1 - \cos \omega_m t) \right] \hat{E}(t) = 0
\]

Parabolic approximation for modulation function

\[
T = 1 - m(1 - \cos \omega_m t) \cong 1 - \frac{1}{2} m(\omega_m t)^2
\]

The solution is then

\[
\hat{E}(t) = E_0 \exp \left( -\frac{t^2}{\tau^2} \right)
\]

with

\[
\tau = \sqrt{\frac{1}{\omega_g \omega_m} \left( \frac{8g}{m} \right)^{1/4}}
\]

Parabolic approximation for pulse peak

\[
\hat{E}(t) = E_0 \left( 1 - \frac{t^2}{\tau^2} \right)
\]

Passing through the modulator yields

\[
\frac{1}{\tau^2} \rightarrow \frac{1}{\tau^2} + \frac{m \omega_m^2}{2}
\]

Giving a pulse shortening rate

\[
\frac{\Delta \tau}{\tau} = \frac{m \omega_m^2 \tau^2}{4}
\]
Slow saturable absorber I

\[
\begin{bmatrix}
g(t) - l_a(t) - l_0 + i \psi + \frac{l_0}{\omega_g^2} \frac{d^2}{dt^2} + t_D \frac{d}{dt}
\end{bmatrix} \hat{E}(t) = 0
\]

nonsaturable loss
saturable loss
saturable gain

Where

\[
l_a(t) = l_i \exp \left( -\sigma_a \int_{-\infty}^{t} |\hat{E}(t')|^2 dt' \right)
\]
\[
g(t) = g_i \exp \left( -\sigma_g \int_{-\infty}^{t} |\hat{E}(t')|^2 dt' \right)
\]

The initial loss and gain are \( l_i \) and \( g_i \), the effective cross-sections are \( \sigma_a \) and \( \sigma_g \).

Expand these to second order in \( \int |\hat{E}(t')|^2 dt' \)

The solution is

\[
\hat{E}(t) = E_0 \text{sech} \left( \frac{t}{\tau} \right)
\]

with

\[
\tau = \frac{1}{\omega_g \sigma_a W} \sqrt{\frac{l_0}{l_i}}
\]

where

\[
W = \int_{-\infty}^{\infty} |\hat{E}(t')|^2 dt'
\]

Increasing
Gain bandwidth
Pulse energy
Saturable absorption
Ease of saturation
Decreases pulse duration

Increase non-saturable loss increases duration
The pulse shortening rate \( t_i(\sigma_{aW})^2 \) due to competition between absorber and finite gain bandwidth \( \rightarrow \) independent of pulsewidth

If we include GVD and self-phase modulation, the solution is a chirped pulse

\[
\hat{E}(t) = E_0 \text{sech}\left(\frac{t}{\tau}\right) \exp\left[i\beta \ln \text{sech}(t/\tau)\right]
\]

Where we assume the SPM is resonant due to the absorber \( \rightarrow \) opposite sign from non-resonant (Kerr) SPM

Resulting relation between dispersion and pulsewidth

Absorption or gain saturation follows the standard saturation curves

\[ I_a = \frac{l_a^0}{1 + I/I_S} \quad \text{and} \quad g = \frac{g_0}{1 + I/I_S} \]

where \( I = |\hat{E}(t)|^2 \)

Using this form, the master equation cannot be solved analytically.

It can be if saturation is linearized

\[ I_a = l_a^0(1 - I/I_S) \quad \text{and} \quad g = g_0(1 - I/I_S) \]

This form can produce solutions of the master equation, but does not provide insight about stability.

To determine if the solution is stable against perturbations, numerics can be used, or more sophisticated theory.
Fast saturable absorber I

\[
\begin{bmatrix}
g - l + i\psi + \frac{g}{\omega_g^2} \frac{d^2}{dt^2} + iD \frac{d^2}{dt^2} + t_D \frac{d}{dt} + (\gamma - i\delta) \hat{E}(t) \hat{E}(t) \end{bmatrix} \hat{E}(t) = 0
\]

The solution to this is also

\[
\hat{E}(t) = E_0 \text{sech}\left(\frac{t}{\tau}\right) \exp[i\beta \ln \text{sech}(t/\tau)]
\]

Plugging this into the master equation yields

No SPM, minimum width at 0 GVD is

\[
\tau_0 = \frac{4g}{\gamma W \omega_g^2}
\]

With SPM, GVD for no chirp is

\[
D = \frac{\delta}{\gamma} \frac{g}{\omega_g^2}
\]

\[
\tau = \frac{4|D|}{\delta W}
\]

A stable pulse requires pulse gain greater than CW gain.

Instability in pure SPM case due to limited gain bandwidth.

The pulse shortening rate is

$$\Delta \tau = \frac{\gamma W}{2 \tau}$$
The fact that the dispersion is not constant turns out to be quite important. Alternating regions of positive and negative dispersion → Causes the pulse to “breath”, stretching and recompressing twice per “period”. The pattern of dispersion variation is called the “dispersion map”.

In a bulk optic laser, the nonlinear coefficient also varies. Non zero in gain medium. Zero elsewhere.

Dispersion management

alters the relationship between power, dispersion and width

Generally need higher power since pulse is stretched much of the time

Allows stable solution in region of normal net dispersion

Changes pulse shape (semi-continuously)

Gaussian in certain cases!

\[ \frac{\gamma P_0 t_0}{\beta_2} = 1 \]
This analysis has been useful for understanding the physical process of pulse formation, but in the ultrabroad band/short pulse limit:

- Higher order effects become very important – particularly dispersion
- Spectral filtering becomes strong (parabolic approximation invalid)
- Self-amplitude modulation saturates